Bijective link between Chapoton's new intervals and bipartite planar maps

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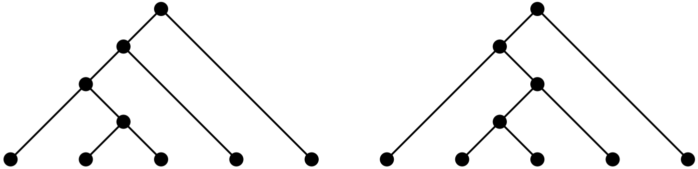
Binary trees

Binary trees : leaves or internal nodes with 2 children

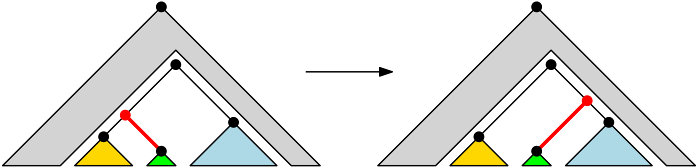


Size : # internal nodes

Enumeration : Catalan numbers $\mathrm{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}$

Rotation on binary trees

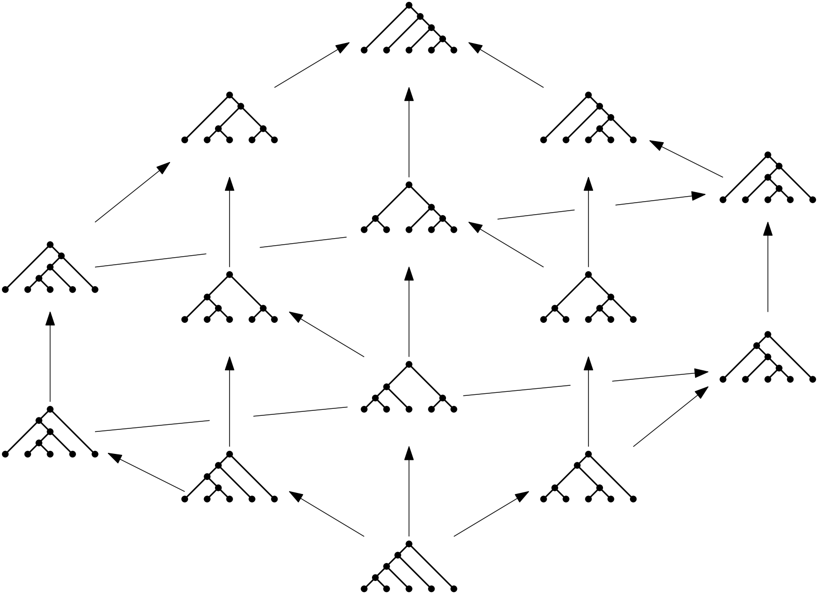
Rotation (from left to right) :



Rotation ⇒ order : Tamari lattice

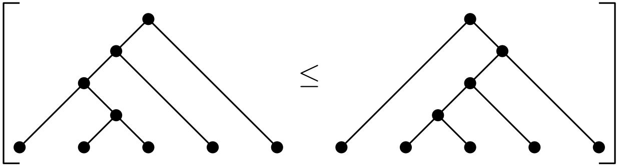
Can also be defined on other Catalan objects (Dyck paths, ...)

Tamari lattice



Tamari intervals

Tamari intervals : a pair of objects $S \leq T$ comparable in Tamari lattice, also denoted $[S, T]$



Counted by Chapoton in 2006 : for all sizes n, the number is

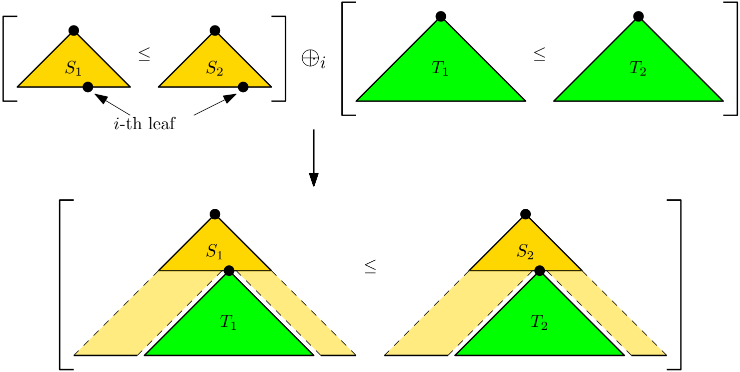
$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$

Same formula as bridgeless planar maps and 3-connected planar triangulations. (There are several bijections.)

How is it done (by Chapoton) ?

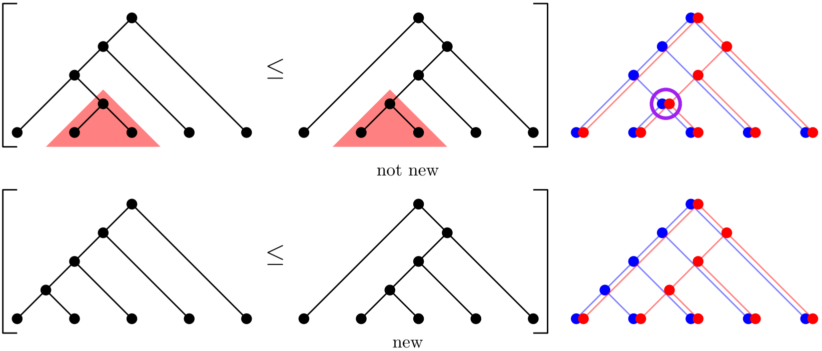
Lego of Tamari intervals

Operation $\oplus_i$: compose two intervals in a big one



New intervals

An interval I is new if it cannot be constructed as $I = I_1 \oplus_i I_2$.



Easy criterion : common non-root internal nodes

Geometrically : new $\Leftrightarrow$ not on the same facet of the associahedron

A structure of operad, with new intervals as atoms

Unique decomposition of general ones into new ones $\Rightarrow$ enumeration

Counting new intervals

Théorème (Chapoton 2006)

The number of new intervals of size n is

$$\frac{3 \cdot 2^{n-2}(2n-2)!}{(n-1)!(n+1)!}.$$

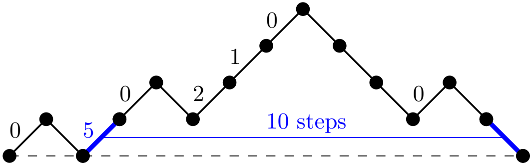
With this formula, Chapoton counted general Tamari intervals.

Same formula as bipartite planar maps!

Dyck paths

Dyck paths :

- Formed by up steps $(1,1)$ and down steps $(1,-1)$,
- Starting and ending on x-axis, while staying above it.



$$V_P = 0, 5, 0, 2, 1, 0, 0$$

Matching steps : connected by horizontal line without obstacle

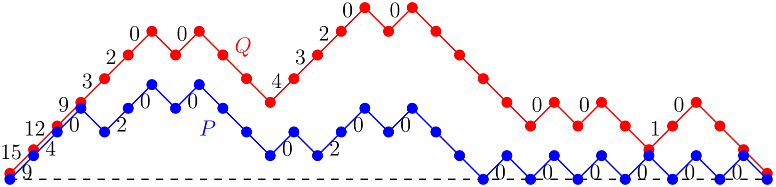
Bracket vector V_P of path P :
$V_P(i) =$ half-length from the i-th up step to its matching down step

Rising contact : up step on x-axis

$\mathbf{rcont}(P)$: number of rising contacts of P.

New intervals, with Dyck paths

Tamari lattice : $P \leq Q \iff V_P \leq V_Q$ componentwise

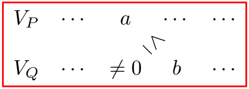


$$V_P = \quad 9, \ \ 4, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0$$

$$V_Q = 15, 12, 9, 3, 2, 0, 0, 4, 3, 2, 0, 0, 0, 0, 1, 0$$

An interval $[P, Q]$ is new iff :

- $V_Q(1) = n$;
- $\forall 1 \leq i \leq n,\ V_Q(i) \neq 0 \Rightarrow V_P(i) \leq V_Q(i+1)$.

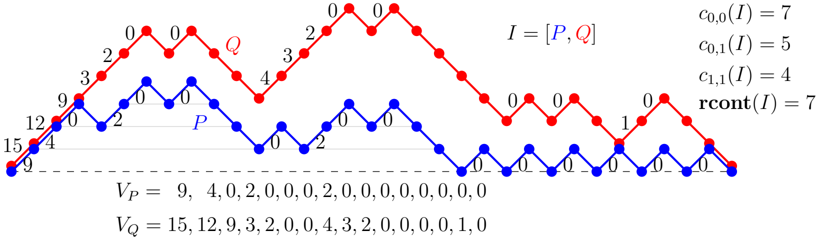


Three statistics (nearly) symmetric

Three statistics on an interval $I = [P, Q]$:

$$\mathbf{c}_{00}(I) = \# \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_{01}(I) = \# \begin{bmatrix} 0 \\ \neq 0 \end{bmatrix}, \quad \mathbf{c}_{11}(I) = \# \begin{bmatrix} \neq 0 \\ \neq 0 \end{bmatrix}.$$

Then $\mathbf{rcont}(I) = \mathbf{rcont}(P)$ (lower path).

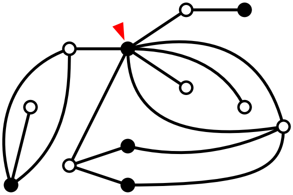


(Experimental) symmetry between $\mathbf{c}_{00}(I), \mathbf{c}_{01}(I), 1 + \mathbf{c}_{11}(I)$ when summing for all new intervals of size n (Chapoton, unpublished)

Similar symmetry in bipartite planar maps. **A link?**

Bipartite planar maps

Bipartite planar map : proper drawing of bipartite graph on the plane, rooted at a corner of a black vertex on the outer face



$\mathbf{white}(M) = 7$

$\mathbf{black}(M) = 5$

$\mathbf{face}(M) = 5$

$\mathbf{outdeg}(M) = 6$

Three statistics of a bipartite planar map M :

$\mathbf{white}(M) = \#$white vertex, $\mathbf{black}(M) = \#$black vertex, $\mathbf{face}(M) = \#$face.

Equidistributed ($\#$ cycles of permutations in $\sigma_\bullet \sigma_\circ \phi = \mathrm{id}_n$)

An auxiliary statistic : $\mathbf{outdeg}(M) =$ half-degree of the outer face

Refined equi-enumeration

Théorème (Chapoton and Fusy, unpublished)

Let $F_{\mathcal{I}}(t,x;u,v,w)$ *be the generating function of new intervals:*

$$F_{\mathcal{I}}(t,x;u,v,w) = \sum_{n\geq 1} t^n \sum_{I\in\mathcal{I}_n} x^{\mathbf{rcont}(I)-1} u^{\mathbf{c}_{00}(I)} v^{\mathbf{c}_{01}(I)} w^{\mathbf{c}_{11}(I)}.$$

Let $F_{\mathcal{M}}(t,x;u,v,w)$ *be the generating function of bipartite planar maps:*

$$F_{\mathcal{M}}(t;u,v,w) = \sum_{n\geq 0} t^n \sum_{M\in\mathcal{M}_n} x^{\mathbf{outdeg}(M)} u^{\mathbf{black}(M)} v^{\mathbf{white}(M)} w^{\mathbf{face}(M)}.$$

Then we have

$$wF_{\mathcal{I}} = tF_{\mathcal{M}}.$$

Proved using recursive decomposition of the two families of objects

A bijective proof ?

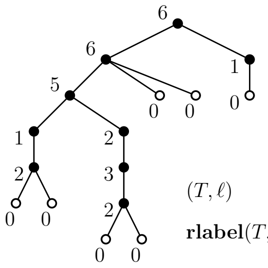
Degree trees

Degree trees : a pair (T, ℓ)

- T: plane tree,
- ℓ: node-labeling on T,

such that, for all node v,

- v is a leaf $\Rightarrow \ell(v) = 0$;
- v has children $v_1, v_2, \ldots, v_k \Rightarrow \ell(v) = k - a + \sum_i \ell(v_i)$ for some $0 \leq a \leq \ell(v_1)$.



(T, ℓ)

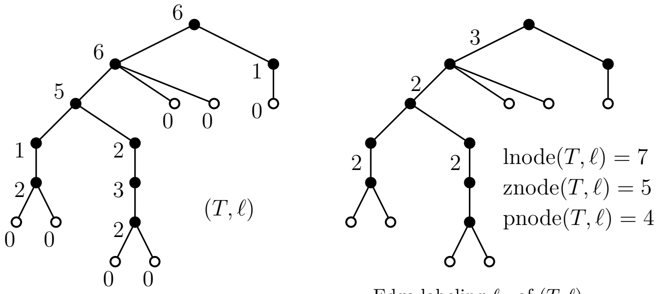
$\textbf{rlabel}(T, \ell) = 6$

$\textbf{rlabel}(T, \ell)$: root label

Degree trees, another version

Edge labeling ℓ_Λ of (T, ℓ): on the leftmost descending edge of each node v, with the value subtracted from $\ell(v)$.

$\ell_\Lambda \Rightarrow \ell$: $\ell(v) = \#$ descendants - sum of edge labels below v

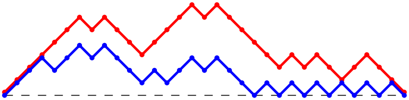


Edge labeling ℓ_Λ of (T, ℓ)

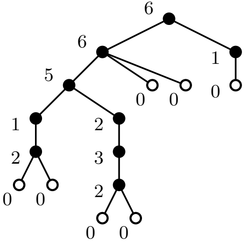
Three statistics :

- **lnode**(T, ℓ): #leaves,
- **znode**(T, ℓ): #nodes with $\ell_\Lambda(e) = 0$ on its leftmost edge e,
- **pnode**(T, ℓ): #nodes with $\ell_\Lambda(e) \neq 0$ on its leftmost edge e.

Bijections

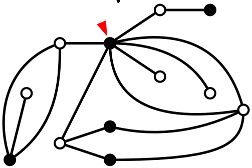


Chapoton's new intervals of size $n+1$



Degree trees with n edges

$$\mathbf{c}_{0,0}(I) = \mathbf{lnode}(T,\ell) = \mathbf{white}(M)$$
$$\mathbf{c}_{0,1}(I) = \mathbf{znode}(T,\ell) = \mathbf{black}(M)$$
$$\mathbf{c}_{1,1}(I) = \mathbf{pnode}(T,\ell) = \mathbf{face}(M) - 1$$
$$\mathbf{rcont}(I) = \mathbf{rlabel}(T,\ell) + 1 = \mathbf{outdeg}(M) + 1$$

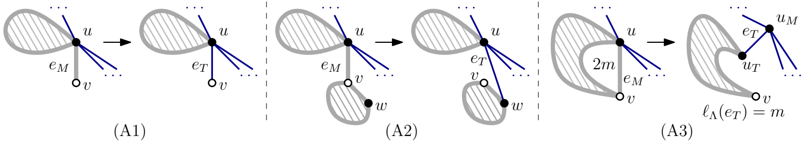


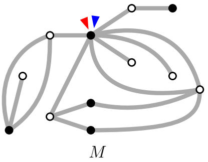
Bipartite planar maps with n edges

From maps to trees : exploration

DFS on edges, clockwise, starting from the root, three rules

edges of M $\rightarrow$ edges of (T, ℓ). Only on black vertices.





Generalizing a bijection of Janson and Stefánsson (2015) on trees

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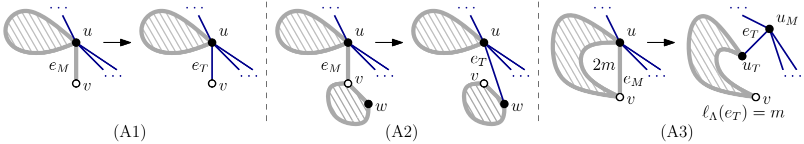

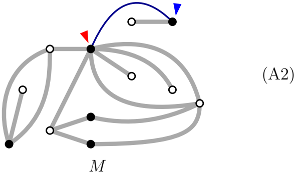

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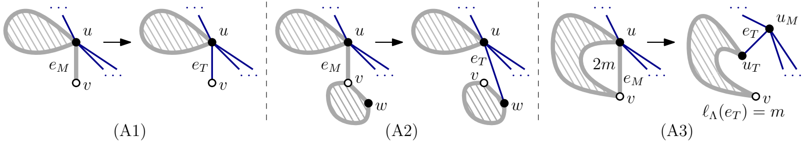

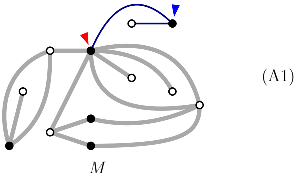

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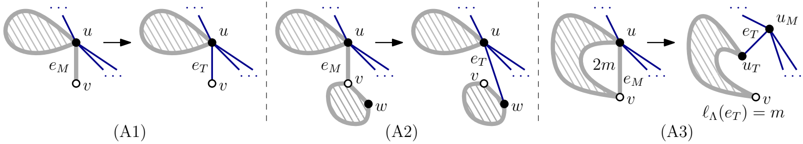

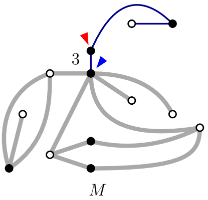

(A3)

Generalizing a bijection of Janson and Stefánsson (2015) on trees

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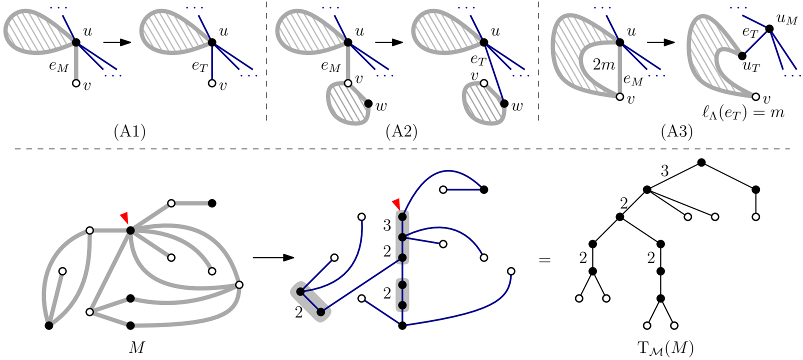
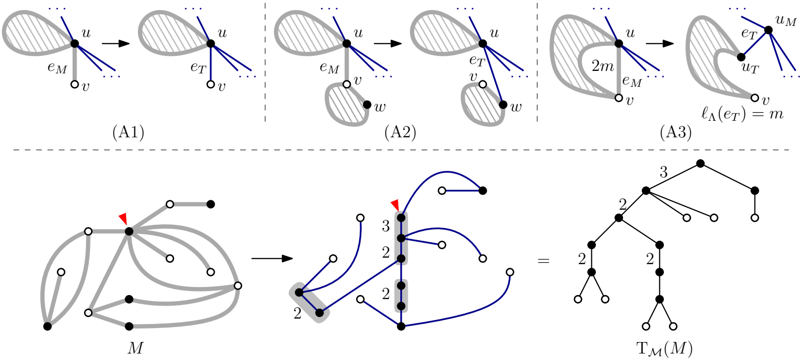


Generalizing a bijection of Janson and Stefánsson (2015) on trees

Correspondence of statistics

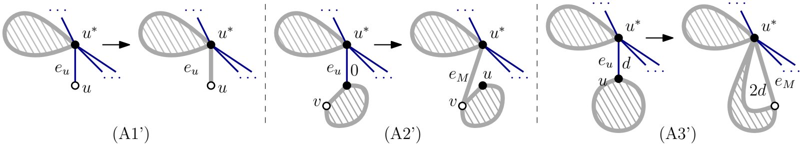


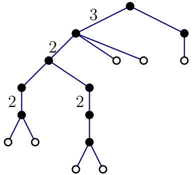
- $\mathbf{white}(M) = \mathbf{lnode}(T, \ell)$: white node ↔ leaves
- $\mathbf{face}(M) = 1 + \mathbf{pnode}(T, \ell)$: inner face ↔ (A3)
- $\mathbf{black}(M) = \mathbf{znode}(T, \ell)$: computation
- $\mathbf{outdeg}(M) = \mathbf{rlabel}(T, \ell)$: inner face ↔ (A3)

From trees to maps: reversed exploration

DFS on edges, counter-clockwise, three rules when exiting an edge

edges of (T, ℓ) $\rightarrow$ edges of M.





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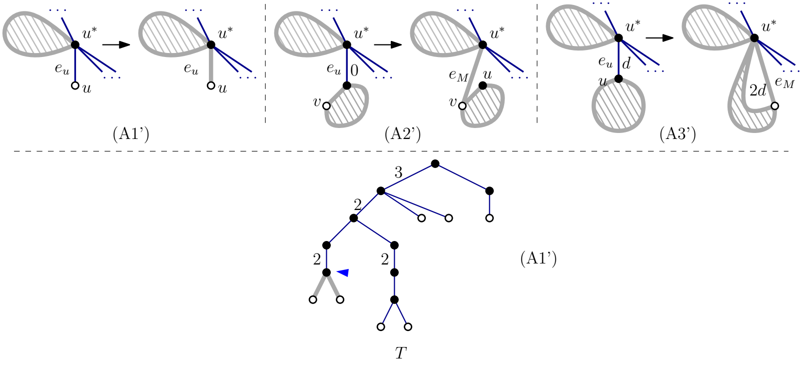

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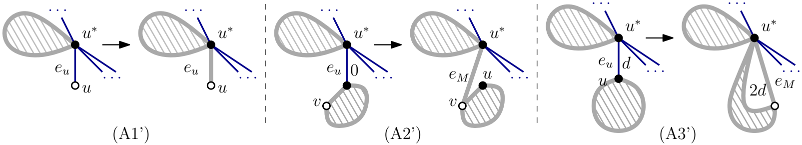

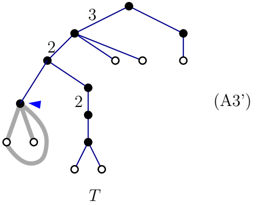

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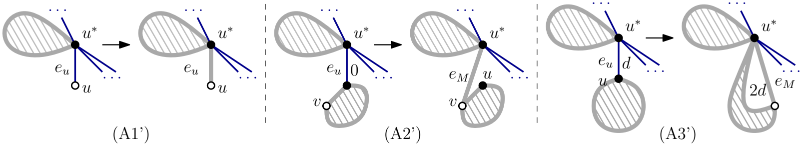

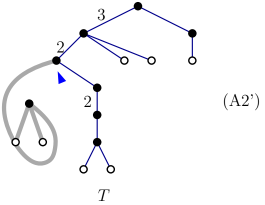

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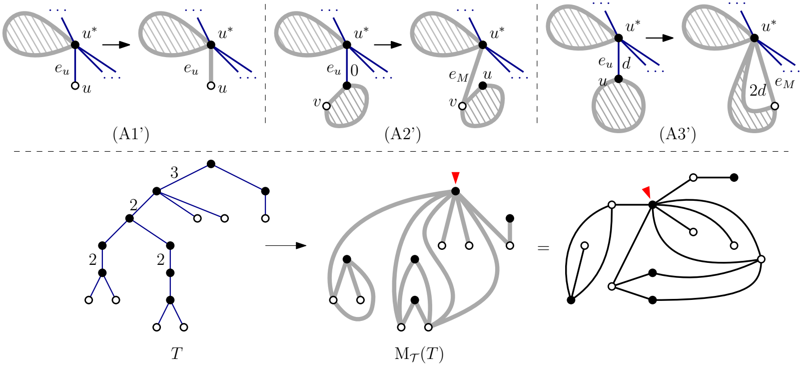
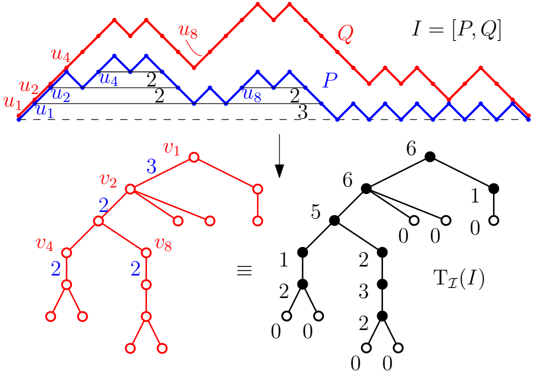


From intervals to trees: contacts counting

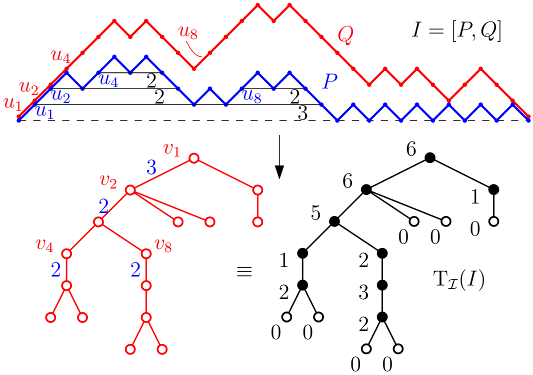
From $I = [P, Q]$ to (T, ℓ) using ℓ_Λ:

- T: from Q' such that $Q = uQ'd$ (as $V_Q(1) = n$)
- i-th up step of $Q \Leftrightarrow i$-th node v_i of T in contour (root included)
- i-th up step of $P \Leftrightarrow$ upward edge of v_{i+1} in T (shift by 1!)
- ℓ_Λ: rising contacts on sub-paths between matching steps

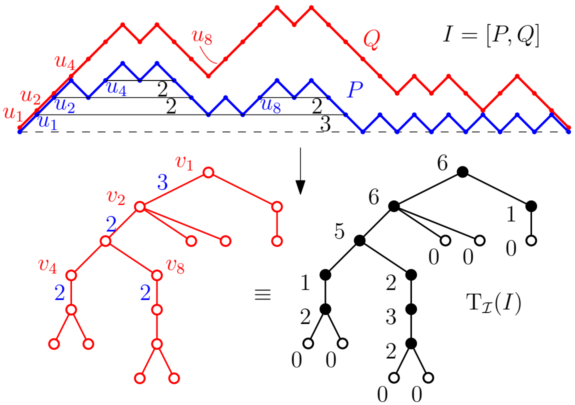


From intervals to trees: correctness

- $V_Q(i)$: # descendants of v_i
- $V_P(i)$: sum of labels of upward edge of v_{i+1} and edges in subtree
- Tamari ⇔ positive vertex label
- New ⇔ label of upward edge of v_{i+1} limited by label of v_{i+1}



Correspondence of statistics

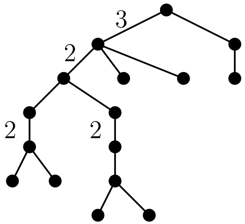


- $\mathbf{c}_{00}(I) = \mathbf{lnode}(T, \ell)$: $V_Q(i) = 0 \Leftrightarrow$ leaf
- $\mathbf{c}_{11}(I) = \mathbf{pnode}(T, \ell)$: $V_P(i) \neq 0 \Leftrightarrow$ non-zero label on edge
- $\mathbf{c}_{01}(I) = \mathbf{znode}(T, \ell)$: computation with size
- $\mathbf{rcont}(I) = \mathbf{rlabel}(T, \ell)$: rising contacts not counted in ℓ_Λ

From trees to intervals: a coloring process

The certificate of a node in (T, ℓ) is defined by a coloring process (reversed prefix order):

- All nodes are black from the start;
- v a leaf $\Rightarrow$ the certificate of v is v itself;
- v not a leaf, with e its leftmost edge $\Rightarrow$ color nodes after v in prefix order in red, stop up to the $(\ell_\Lambda(e) + 1)$-st black node. The last node visited is the certificate of v.



Certificate function c of (T, ℓ): $c(u) = \#$nodes whose certificate is u

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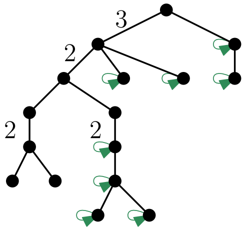

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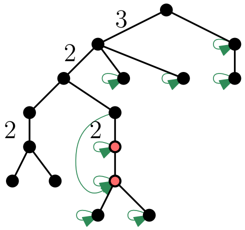

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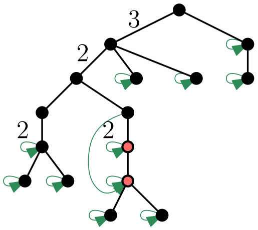

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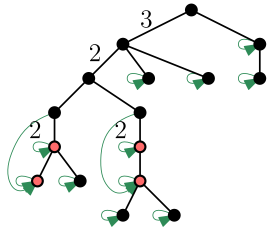

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From trees to intervals: a coloring process

The certificate of a node in (T, ℓ) is defined by a coloring process (reversed prefix order):

- All nodes are black from the start;
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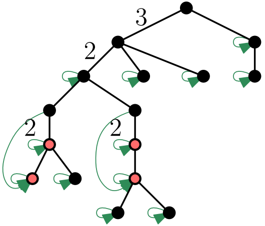

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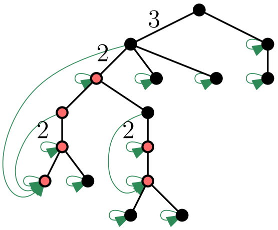

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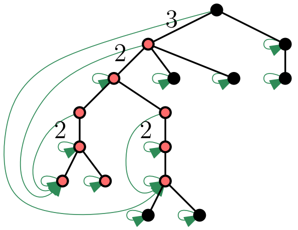

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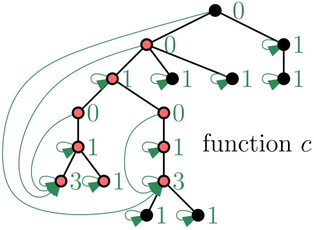

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From trees to intervals: certificate function

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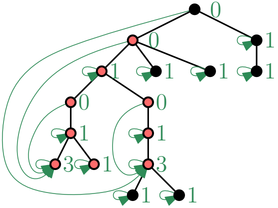
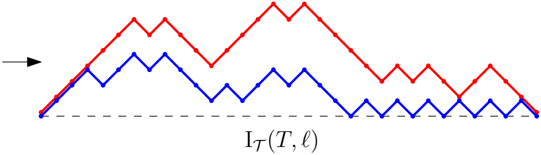
From (T, ℓ) to $I = [P, Q]$:

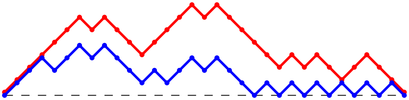
- P: concatenation of $ud^{c(v)}$ for all v in prefix order;
- Q: $uQ'd$ with Q' obtained from the contour walk of T.



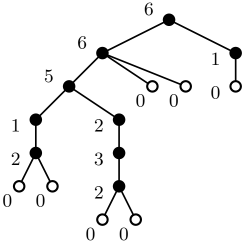
function c



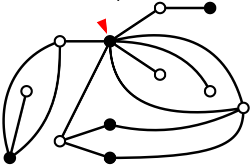
Recapitulation



Chapoton's new intervals of size $n+1$



Degree trees with n edges

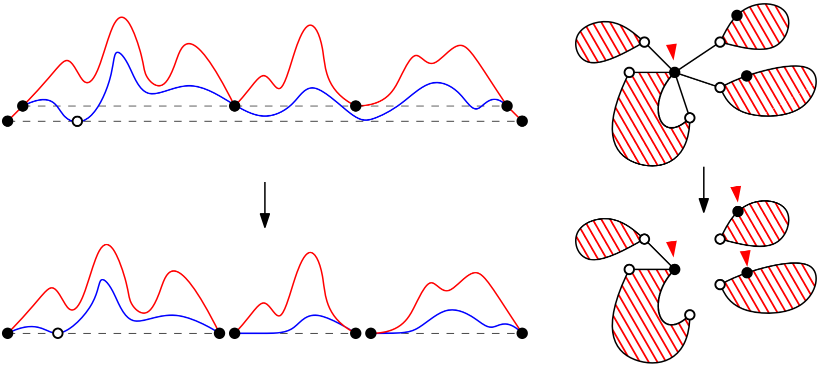


Bipartite planar maps with n edges

$$\mathbf{c}_{0,0}(I) = \mathbf{lnode}(T,\ell) = \mathbf{white}(M)$$
$$\mathbf{c}_{0,1}(I) = \mathbf{znode}(T,\ell) = \mathbf{black}(M)$$
$$\mathbf{c}_{1,1}(I) = \mathbf{pnode}(T,\ell) = \mathbf{face}(M) - 1$$
$$\mathbf{rcont}(I) = \mathbf{rlabel}(T,\ell) + 1 = \mathbf{outdeg}(M) + 1$$

What is really happening

Recursive decomposition of the two families of objects (Chapoton and Fusy, unpublished):



Degree tree is in fact the decomposition tree.

The bijections are all canonical w.r.t. these decompositions.

Work in progress (?)

- $\mathbb{S}_3$ symmetry for bipartite maps, how about new intervals?
- At least one explained: white $\leftrightarrow$ face $\Leftrightarrow$ duality of intervals
- Relation with $\beta(0,1)$-trees ? And other objects ?
- Recent new direct bijection between degree trees and linear planar 3-connected λ-terms (arXiv:2202.03542)
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Thank you for listening!