Compacted binary trees, stretched exponential and asymptotic behavior of recurrences

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Joint work with Andrew Elvey Price and Michael Wallner

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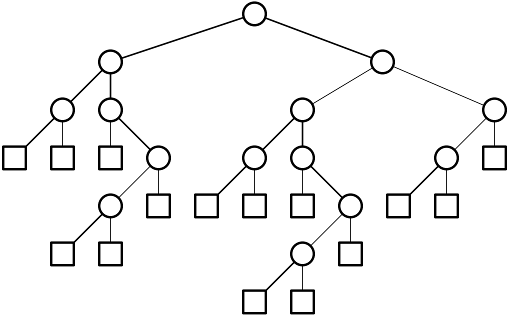
What is this talk about?

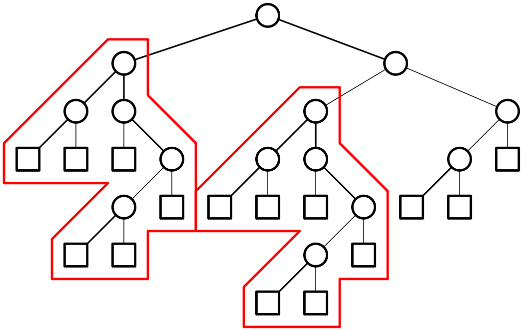
- A "new method" to get asymptotic behavior of certain recurrences
- ... without generating function (*gasp!*)
- ... illustrated with compacted trees as example
- ... and some progress for generalization.

Compacting binary trees



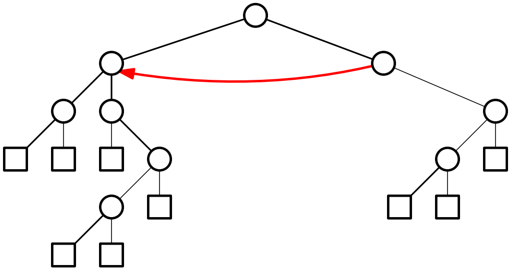
We try to compress a binary tree ...

Compacting binary trees



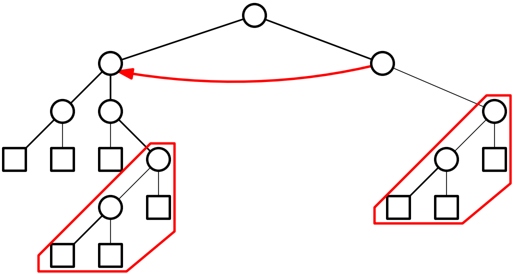
... by finding identical sub-trees ...

Compacting binary trees



... and storing them only once ...

Compacting binary trees



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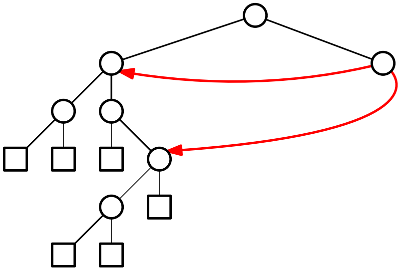
Compacting binary trees



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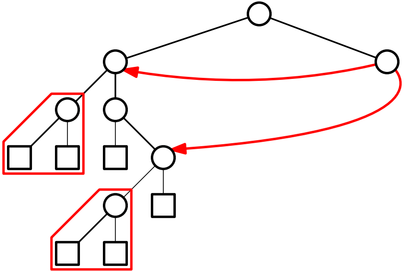
Compacting binary trees



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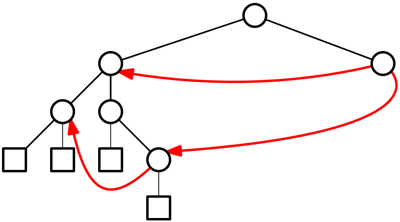
Compacting binary trees



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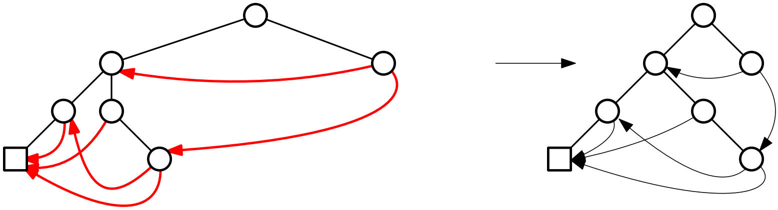
Compacting binary trees



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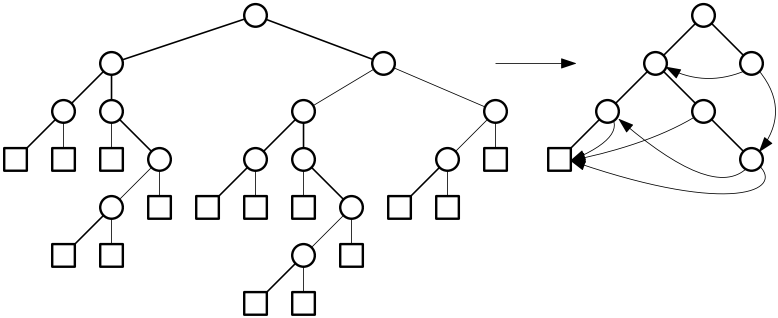
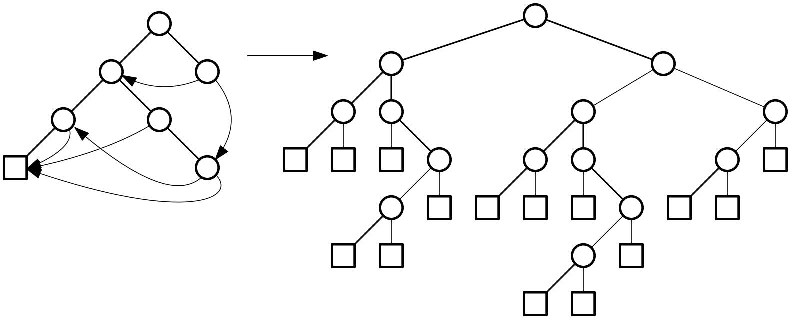
Compacting binary trees



The **compacted trees** are trees with pointers obtained in this way.

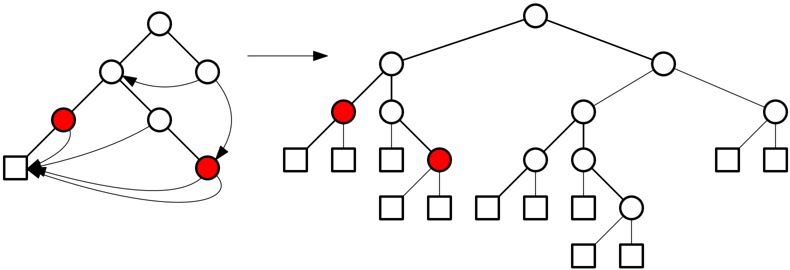
Compacted trees



A **compacted tree** is a binary tree such that

- every leaf (except the first one) is a pointer ...
- ... towards a node preceding it in postfix order,
- and each node has a distinct "decompressed" sub-tree.

Relaxed trees



A **relaxed tree** is a binary tree such that

- every leaf (except the first one) is a pointer ...
- ... towards a node preceding it in postfix order,
- ~~and each node has a distinct "decompressed" sub-tree.~~

What we know, and what we want to know

- (Flajolet, Sipala, Steyaert 1990)
  - Linear algorithm to "compactify" a binary tree of size n
  - Average size of the compacted tree : $O(n/\log n)$
- (Genitrini, Gittenberger, Kauers, Wallner 2019)
  n nodes, with right height $\leq k$
  - Relaxed trees :

$$\gamma_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)\right)^n n^{-k/2}$$

  - compacted trees :

$$\gamma_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)\right)^n n^{-\frac{k}{2}-\frac{1}{k+3}-\left(\frac{1}{4}-\frac{1}{k+3}\right)\frac{1}{\cos^2\left(\frac{\pi}{k+3}\right)}}$$

And without any restrictions?

Our result

- c_n : the number of compacted trees with n nodes
- r_n : the number of relaxed trees with n nodes

Theorem (Elvey Price, F., Wallner 2021)

When $n \to \infty$, we have

$$c_n = \Theta\left(n! 4^n e^{3a_1 n^{1/3}} n^{3/4}\right), \qquad r_n = \Theta\left(n! 4^n e^{3a_1 n^{1/3}} n\right).$$

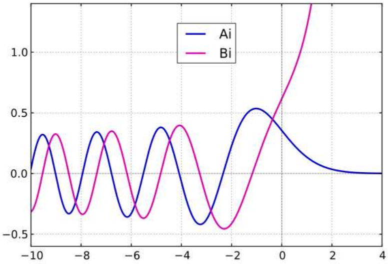
Here, a_1 is the largest root of the Airy function $\mathrm{Ai}(x)$, solution of $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ with $\mathrm{Ai}(x) \to 0$ when $x \to +\infty$.

We don't have the multiplicative constant !

Stretched exponential: $e^{3a_1 n^{1/3}}$

Probability for a relaxed tree of size n to be compacted : $\Theta(n^{-1/4})$.

How do we do that ?

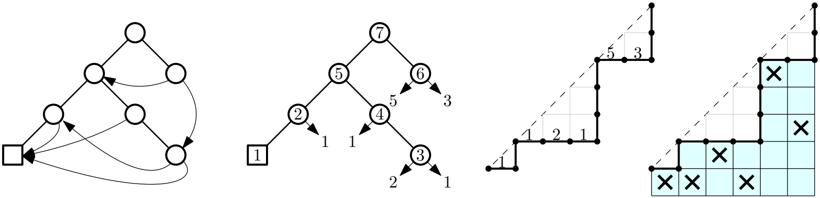


- Bijection with decorated Dyck paths
- Recurrence with two parameters
- Heuristics for typical behaviors
- Truncation of the heuristics ⇒ proof of the bounds

Solely based on the recurrence, the method is relatively simple.

Encoding by decorated Dyck paths (relaxed version)

First we deal with relaxed trees:

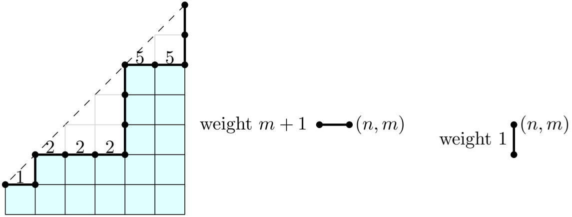


From relaxed tree to decorated Dyck paths:

- Label the nodes in postfix order, detach the pointers
- Draw the Dyck path : $\rightarrow$ for pointer, $\uparrow$ for finishing a node
- Put pointer labels on horizontal steps

A recurrence for relaxed trees

Weight $m+1$ for step $\rightarrow$ on height m.



Proposition

Let $r_{n,m}$ be the weighted sum of paths ending at (n,m). Then

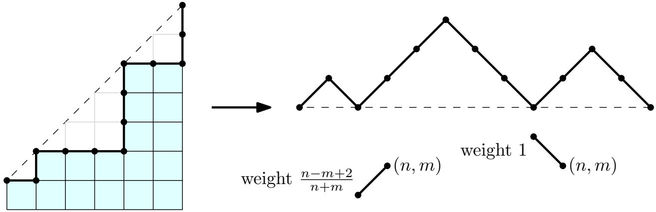
$$r_{n,m} = (m+1)r_{n-1,m} + r_{n,m-1}, \quad \text{for } n \geq m \geq 1,$$
$$r_{n,m} = 0, \quad \text{for } n < m,$$
$$r_{n,0} = 1, \quad \text{for } n \geq 0.$$

The number of relaxed trees with n nodes is $r_{n,n}$.

A transformation

Change of coordinates: $(n, m) \to (n+m, n-m)$

We take $d_{n+m,n-m} = r_{n,m}/n!$, as labeled structure.



Recurrence :

$$d_{n,m} = \frac{n-m+2}{n+m} d_{n-1,m-1} + d_{n-1,m+1}$$

The number of size n relaxed trees: $r_n = n! d_{2n,0}$.

Some observations

$$d_{n,m} = \frac{n-m+2}{n+m} d_{n-1,m-1} + d_{n-1,m+1}$$

Recurrence ⇒ diff. eq. in two variables, hard to solve.

Numerical observations :

$$d_{2n,0} = \Theta\left(4^n \rho^{n^{1/3}} n\right)$$

- 4^n from Dyck paths.
- Why a stretched exponential?

A higher up step has a lower weight!

A first heuristics

Consider Dyck paths of length $2n$ and maximal height $\leq n^{\alpha}$, $\alpha < 1/2$.

Proposition (Kousha 2012)

A uniformly random path has height n^{α} ($\alpha < 1/2$) with probability

$$\log(\mathbb{P}[\text{height} \leq n^{\alpha}]) \sim -\pi^2 n^{1-2\alpha}$$

Weight of a typical up step:

$$\frac{\Theta(n) - \Theta(n^{\alpha})}{\Theta(n) + \Theta(n^{\alpha})} = 1 - \Theta(n^{\alpha-1}).$$

Typically $\Theta(n)$ such steps, thus a total weight

$$\left(1 - \Theta(n^{\alpha-1})\right)^{\Theta(n)} = \exp\left(-\Theta(n^{\alpha})\right).$$

Total contribution

$$\exp\left(-\Theta(n^{\alpha}) - \Theta(n^{1-2\alpha})\right),$$

maximized at $\alpha = 1/3$, giving a stretched exponential $\exp(-\Theta(n^{1/3}))$.

The correct scaling

Too heuristic... But this shows that the correct height is $n^{1/3}$!

Ansatz:

$$d_{n,m} \sim h(n) f(n^{-1/3}(m+1)),$$
$$s(n) = \frac{h(n)}{h(n-1)} = 2 + cn^{-2/3} + O(n^{-1}).$$

- $h(n)$: general growth in n, around $2^n \rho^{n^{1/3}}$ for some ρ
- $f(x)$: scaling with typical height $n^{1/3}$

Suppose that $m = \kappa n^{1/3} - 1$.

Ansatz + recurrence :

$$f(\kappa)s(n) = \frac{n - \kappa n^{1/3} + 1}{n + \kappa n^{1/3} - 1} f\left(\frac{\kappa n^{1/3} - 2}{(n-1)^{1/3}}\right) + f\left(\frac{\kappa n^{1/3}}{(n-1)^{1/3}}\right).$$

Approximately,

$$0 = (c + 2\kappa) f(\kappa) - f''(\kappa) + O(n^{-1/3}).$$

The first estimation

$$0 = (c + 2\kappa) f(\kappa) - f''(\kappa) + O(n^{-1/3}).$$

Roughly the equation of the Airy function !

As $f(\kappa) \to 0$ for $\kappa \to \infty$, we have

$$f(\kappa) \approx b\mathrm{Ai}\left(\frac{c + 2\kappa}{2^{2/3}}\right).$$

$f(\kappa) \to 0$ for $\kappa \to 0 \Rightarrow c = 2^{2/3} a_1$.

Asymptotic behavior of $\mathrm{Ai}(x)$ near $x \to a_1$ implies

$$r_n = n! d_{2n,0} = n! 4^n \exp\left(3 a_1 n^{1/3} + \dots\right).$$

Refined heuristics

Ansatz of order 2 :

$$d_{n,m} \sim h(n) \left(f(n^{-1/3}(m+1)) + n^{-1/3} g(n^{-1/3}(m+1)) \right),$$
$$s(n) = 2 + cn^{-2/3} + dn^{-1} + O(n^{-4/3}).$$

We get the polynomial term:

$$r_n = n! d_{2n,0} \approx n! 4^n \exp\left(3a_1 n^{1/3}\right) n.$$

Ansatz in general :

$$d_{n,m} \approx h(n) \sum_{j=0}^{k} f_j(n^{-1/3}(m+1)) n^{-j/3},$$
$$s(n) = 2 + \gamma_2 n^{-2/3} + \gamma_3 n^{-1} + \ldots + \gamma_k n^{-k/3} + o(n^{-k/3}).$$

A truncation suffices, but still heuristics.

Sandwiching the asymptotics

If there are positive $(s_n)_{n\geq 1}$ and $(X_{n,m})_{n\geq m\geq 0}$ such that

$$X_{n,m}s_n \leq \frac{n-m+2}{n+m}X_{n-1,m-1} + X_{n-1,m+1},$$

for all m for large enough n.

Let $h_n = \prod_{i=1}^{n} s_n$, then $X_{n,m}h_n \leq b_0 d_{n,m}$ for some constant b_0.

Lower bound!

Reversing the inequality give an upper bound!

Lower bound - *ansatz* and expansion

We take

$$X_{n,m} = \left(1 - \frac{2m^2}{3n} + \frac{m}{2n}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right),$$
$$s_n = 2 + \frac{2^{2/3}a_1}{n^{2/3}} + \frac{8}{3n} - \frac{1}{n^{7/6}}.$$

The difference is

$$P_{n,m} = -X_{n,m}s_n + \frac{n-m+2}{n+m}X_{n-1,m-1} + X_{n-1,m+1}.$$

Only need to prove $P_{n,m} \geq 0$ for $m < n^{2/3-\varepsilon}$. The other zone negligible.

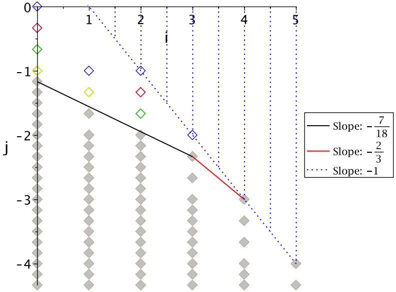
By substitution and asymptotic expansion near n, we have

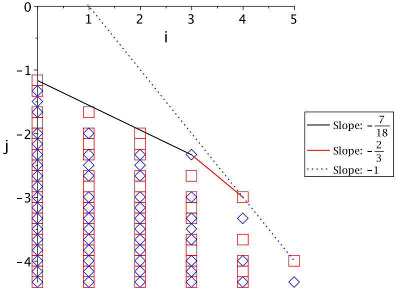
$$P_{n,m} = p_0(n,m)\mathrm{Ai}(\alpha) + p_1(n,m)\mathrm{Ai}'(\alpha), \text{ with } \alpha = a_1 + \frac{2^{1/3}m}{n^{1/3}}.$$

$p_0(n,m), p_1(n,m)$: series in $n^{-1/6}$ with polynomial coeffs in m.

Lower bound - Newton polygon

$$P_{n,m} = \mathrm{Ai}(\alpha)\left(\frac{1}{n^{7/6}} - \frac{2^{5/3}a_1 m}{3n^{5/3}} - \frac{41m^2}{9n^2} - \frac{2^{8/3}a_1 m^3}{3n^{8/3}} - \frac{34m^4}{9n^3} + \dots\right) + \mathrm{Ai}'(\alpha)\left(\frac{2^{1/3}}{n^{3/2}} - \frac{8a_1 m}{9n^2} - 19\frac{2^{1/3}m^2}{9n^{7/3}} - \frac{2^{13/3}m^3}{9n^{7/3}} + \dots\right).$$





Lower bound - case analysis

$$P_{n,m} = \mathrm{Ai}(\alpha)\left(\frac{1}{n^{7/6}} - \frac{2^{5/3}a_1 m}{3n^{5/3}} - \frac{41m^2}{9n^2} - \frac{2^{8/3}a_1 m^3}{3n^{8/3}} - \frac{34m^4}{9n^3} + \dots\right) + \mathrm{Ai}'(\alpha)\left(\frac{2^{1/3}}{n^{3/2}} - \frac{8a_1 m}{9n^2} - 19\frac{2^{1/3}m^2}{9n^{7/3}} - \frac{2^{13/3}m^3}{9n^{7/3}} + \dots\right).$$

- $m \leq x_0(n/2)^{1/3}$, where $\mathrm{Ai}'(a_1 + x)$ changes sign,
- $x_0(n/2)^{1/3} < m \leq n^{7/18}$,
- $n^{7/18} < m < n^{2/3-\varepsilon}$.

All cases are positive using properties of the Airy function.

Upper bound

It is the same, with a different *ansatz*:

$$\hat{X}_{n,m} = \left(1 - \frac{2m^2}{3n} + \frac{m}{2n} + \frac{3m^4}{10n^2}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right),$$

$$\hat{s}_n = 2 + \frac{2^{2/3}a_1}{n^{2/3}} + \frac{8}{3n} + \frac{1}{n^{7/6}}.$$

Yet another case analysis ...

$$r_n = \Theta\left(n! 4^n e^{3a_1 n^{1/3}} n\right).$$

Cherry lemma

On compacted trees:

Lemma

For a relaxed tree T, if no cherry  *reproduces a node that has appeared, then T is compacted.*

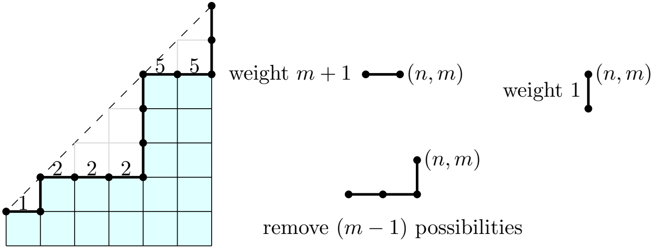
T not compacted $\Rightarrow$ two nodes with the same decompressed trees

The same holds for their children.

Descend until reaching a cherry

Encoding by decorated Dyck paths (compacted version)

Cherry lemma  ⇒ avoid certain 



Proposition

Let $e_{n,m}$ be the number of "strict" decorated paths to (n,m). Then

$$e_{n,m} = (m+1)e_{n-1,m} + e_{n,m-1} - (m-1)e_{n-2,m-1}, \text{ for } n \geq m \geq 1.$$

The number of compacted trees with n nodes is $c_n = e_{n,n}$.

Compacted trees

Recurrence for compacted trees:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{2(n-m-2)}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Negative terms ...

Sandwich it by two positive recurrences.

With two appropriate *Ansätze*, we have

$$c_n = \Theta\left(n! 4^n e^{3a_1 n^{1/3}} n^{3/4}\right).$$

A change in the polynomial factor

Ansatz for lower bound :

$$\hat{X}_{n,m} = \left(1 - \frac{2m^2}{3n} + \frac{m}{4n}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right),$$

$$\hat{s}_n = 2 + \frac{2^{2/3}a_1}{n^{2/3}} + \frac{13}{6n} - \frac{1}{n^{7/6}}.$$

The only difference in $\hat{s}_n \Rightarrow$ change the polynomial factor

An application on automata

Theorem (Elvey Price, F., Wallner 2020)

The number $m_{2,n}$ of minimal automata for finite languages in $A = \{a, b\}$ with n states is

$$m_{2,n} = \Theta\left(n! 8^n e^{3a_1 n^{1/3}} n^{7/8}\right).$$

- Similar "compression": minimal automata as compressed trie
- Encoding by decorated Dyck paths, similar recurrence
- A "cherry lemma"
- Exactly the same method, can do any fixed alphabet size

Summing up

What is good:

- Using only a (quite simple) recurrence;
- Without looking at the generating function;
- Relatively simple, so possible to generalize.
- Sometimes negative terms are not a problem.

Still need work :

- Which type of recurrence? Which type of diff. eq.?
- We still need to start from some heuristics...
- And we miss the multiplicative constant.

Already some other applications!

Michael Fuchs, Guan-Ru Yu, Louxin Zhang, *On the Asymptotic Growth of the Number of Tree-Child Networks*, European J. Combin., 2021.

Yu-Sheng Chang, Michael Fuchs, Hexuan Liu, Michael Wallner, Guan-Ru Yu, *Enumerative and Distributional Results for d-combining Tree-Child Networks*, arXiv:2209.03850, 2022.

Ongoing work

- With Baptiste Louf, we are trying to apply the method to maps.
- Classification of "linearly rational up-step" recurrences:
  - degenerated or trivial,
  - stretched exponential $\rho^{n^{1/3}}$,
  - macroscopic limit,
  - ... maybe more?
- General theorem for stretched exponential other than the Airy type
  - Whittaker type: $\rho^{n^{1/2}}$,
  - ... and further types like $\rho^{n^{\frac{p}{p+2}}}$.

Any recurrences in two parameters for asymptotics?

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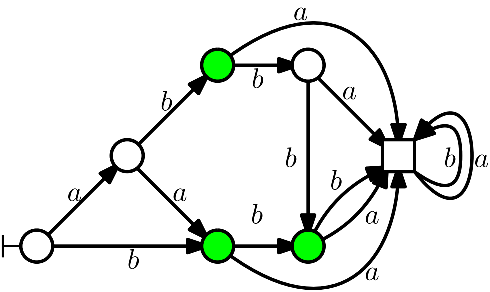
Any recurrences in two parameters for asymptotics?

Thank you for your attention!

Automata



Deterministic automaton Q on alphabet A:

- States and transitions,
- Initial state q_0 and some final states,
- Recognizing $w \Leftrightarrow$ the walk from q_0 reading w arrives at a final state.

Example: aab recognized, but $aaba$ not.

Minimal automata of a finite language

A language = a set of words ⇒ a unique minimal automaton

An automaton is

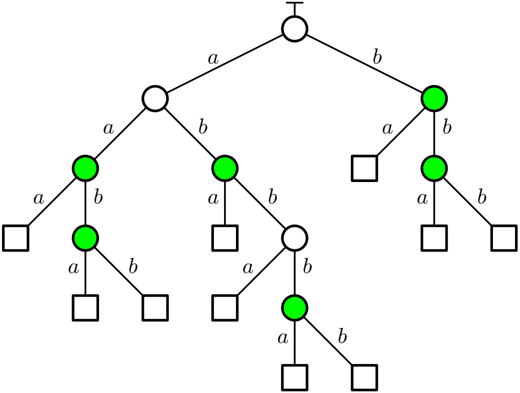
- accessible: all states reachable from the initial one,
- acyclic: no oriented cycle,
- reduced: no redundant state for language recognition.

These three conditions ⇔ minimal automaton of some finite language

Question : How many such automata with n states?

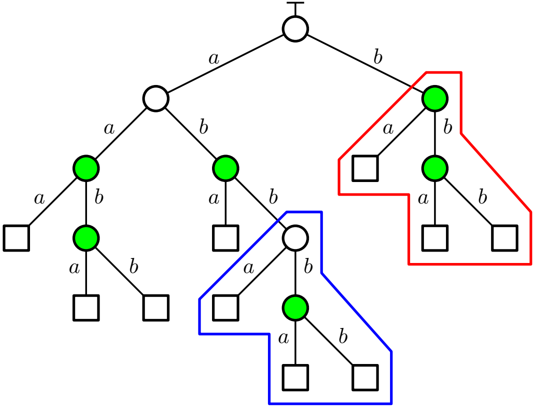
Quite “compacted trees”!

Minimize a trie



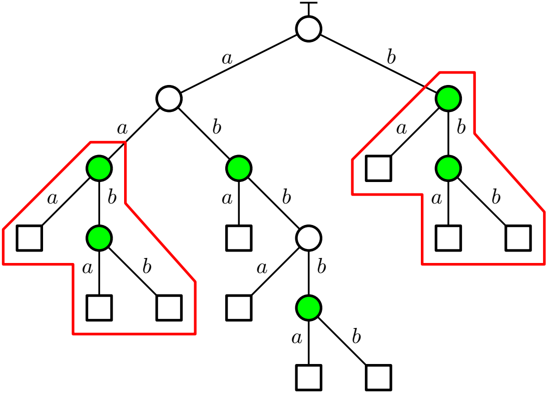
Take a trie and compactify it ...

Minimize a trie



... with sub-trees with identical coloring ...

Minimize a trie



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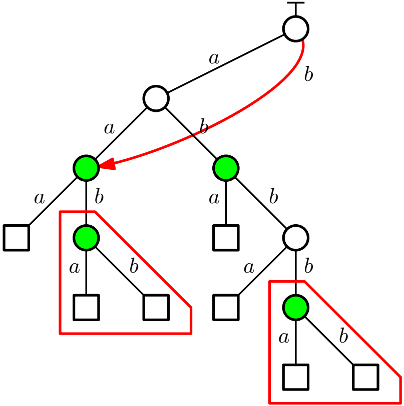
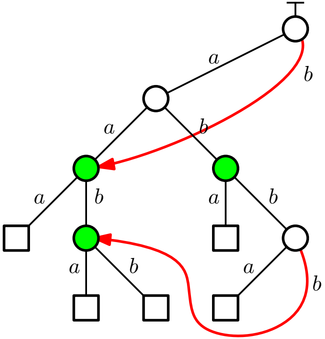


... while exhausting all possibilities ...

Minimize a trie



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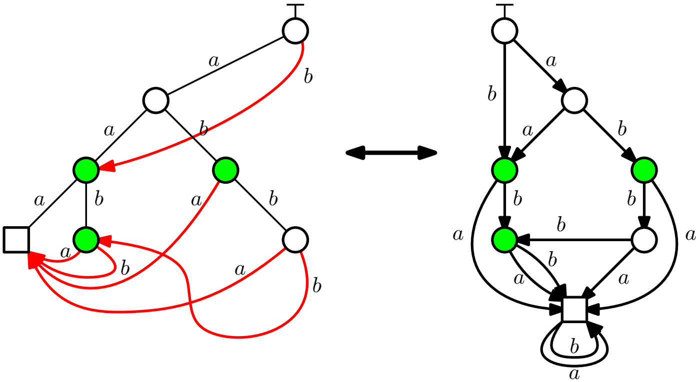
Minimize a trie



... and we get a minimal automaton. > Back <