Steep-bounce zeta map in the parabolic Cataland

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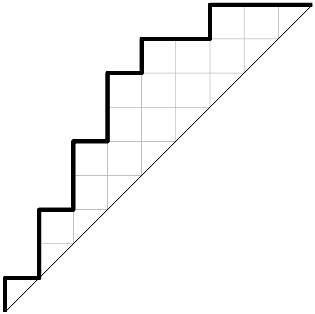
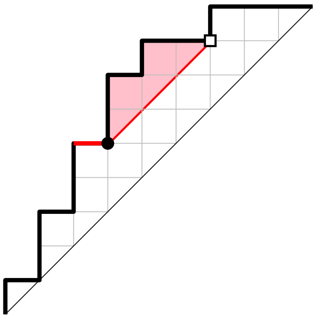
Tamari lattice, as an order on Dyck paths



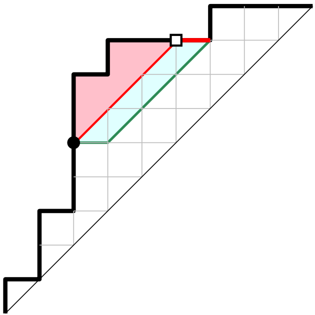
Dyck path : n north steps (N) and n east steps (E), above the diagonal. Counted by Catalan numbers

Tamari lattice, as an order on Dyck paths



Covering relation: take a valley •, let □ be the next point wiht the same distance to the diagonal...

Tamari lattice, as an order on Dyck paths

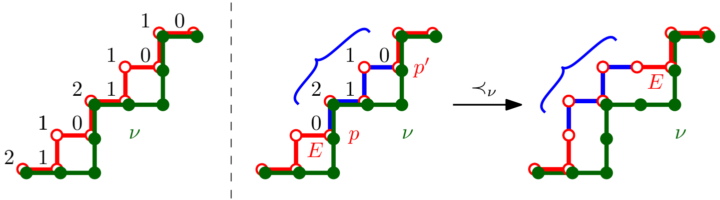


..., and push the segment to the left. The path gets larger. This gives the **Tamari lattice**.

ν-Tamari lattice

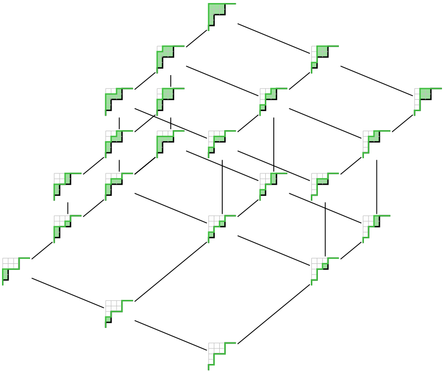
Generalization with ν an arbitrary directed walk as "diagonal"!

Horizontal distance = # east steps until touching the other side of ν



ν-Tamari lattice (Préville-Ratelle and Viennot 2014): $\mathcal{T}_\nu$ with arbitrary ν (called canopy) with steps N, E.

Why is it important ?



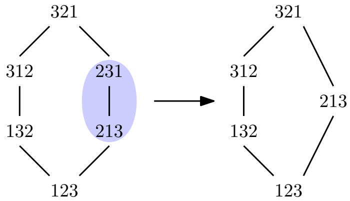
- Generalizing a lot of cases (m-Tamari, rational Tamari)
- Bijective links (non-separable planar maps and related objects)
- Algebraic aspect (subword complexes, Diagonal coinvariant spaces, *etc.*)

Tamari lattice, as quotient of the weak order

$\mathfrak{S}_n$ as a Coxeter group generated by $s_i = (i, i+1)$

For $w \in \mathfrak{S}_n$, $\ell(w) =$ min. length of factorization of w into s_i's.

Weak order : w covered by w' iff $w' = ws_i$ and $\ell(w') = \ell(w) + 1$



Sylvester class : permutations with the same binary search tree

Only one 231-avoiding in each class. Induced order = Tamari.

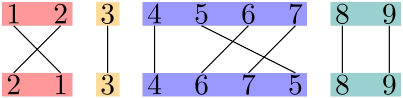
Works for other types

Parabolic subgroup and parabolic quotient of $\mathfrak{S}_n$

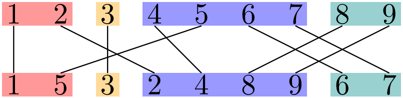
Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n.

Parabolic subgroup : $\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k} \subset \mathfrak{S}_n$.

Generated by s_i except for $i = \alpha_1 + \alpha_2 + \cdots + \alpha_j$.



Parabolic quotient : $\mathfrak{S}_n^\alpha = \mathfrak{S}_n / (\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k})$.



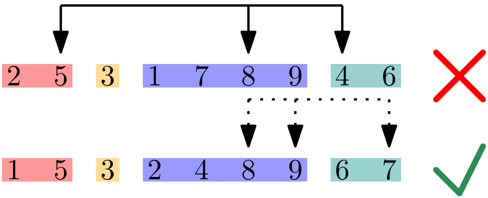
Increasing order in each block

Parabolic permutations avoiding 231

Pattern $(\alpha, 231)$: three indices $i < j < k$ in three distinct blocks with

- $w(k) < w(i) < w(j)$,
- $w(k) + 1 = w(i)$.

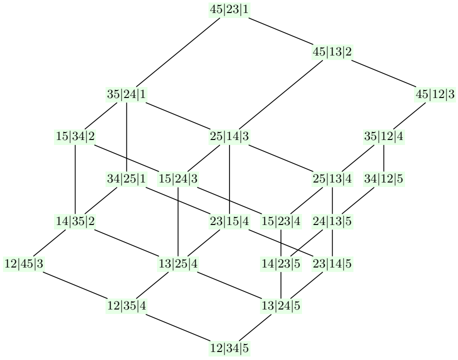
$(\alpha, 231)$-avoiding permutations: without $(\alpha, 231)$ patterns



$\mathfrak{S}_n^\alpha(231)$: set of $(\alpha, 231)$-avoiding permutations

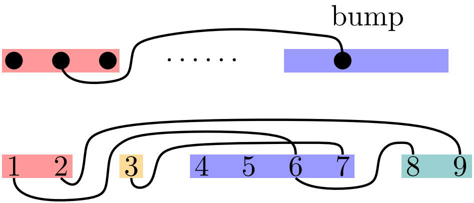
Parabolic Tamari lattice

Parabolic Tamari lattice $\mathcal{T}_n^\alpha$ = weak order restricted to $\mathfrak{S}_n^\alpha(231)$ (Mühle and Williams 2018+)



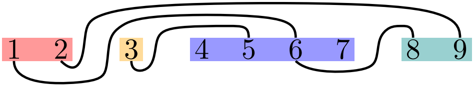
Works for other types!

Parabolic non-crossing partitions



$\{1,6,8\},\{2,9\},\{3,7\},\{4\},\{5\}$

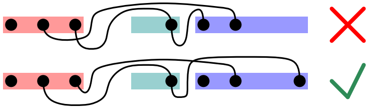
Parabolic α-partition: a set of bumps, ≤ 1 incoming/outgoing



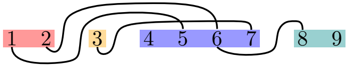
Parabolic non-crossing α-partition : without bumps crossing

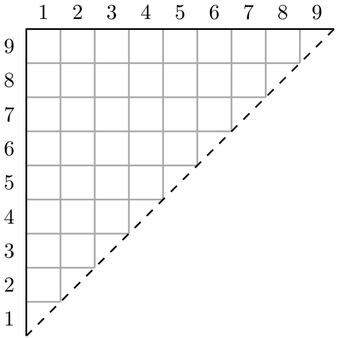
Parabolic non-nesting partitions

Parabolic non-nesting α-partition : no bumps $(i, j), (k, \ell)$ with $i < k < \ell < j$.



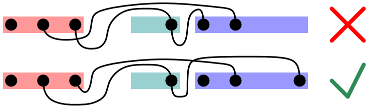
Encoding with points (i, j)





Parabolic non-nesting partitions

Parabolic non-nesting α-partition : no bumps $(i,j),(k,\ell)$ with $i<k<\ell<j$.



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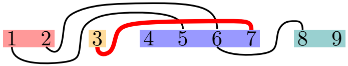

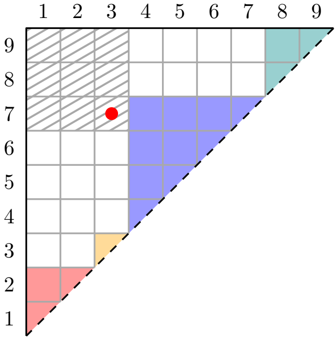

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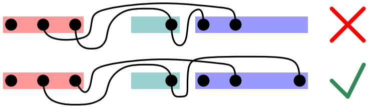

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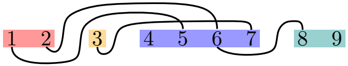

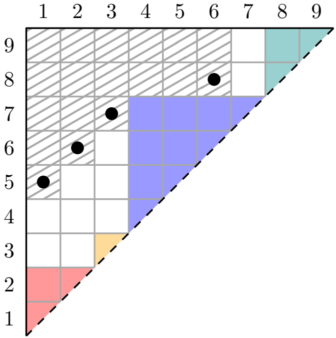

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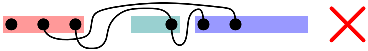
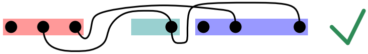
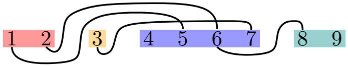
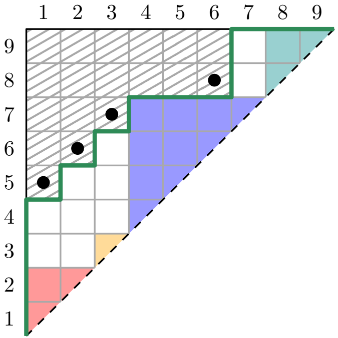




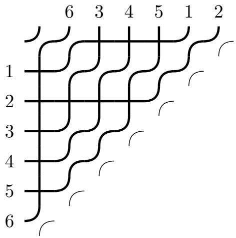
Bounce pair: A Dyck path above a bounce path





Detour to pipe dreams

Hopf algebra on pipe dreams (Bergeron, Ceballos et Pilaud, 2018+).



Dim. of homogeneous comps. of a sub-algebra (generated by identities) = # pipe dreams with an "identity by block" permutation

Proposition (Bergeron, Ceballos and Pilaud, 2018+)

Pipe dreams whose permutation is an "identity by block" of size n are in bijection with bounce pairs of order n.

Already a link to the parabolic Catalan objects!

Counting and relations ?

- All three objects are in bijection (Mühle and Williams), but not easy.
- Numbers of parabolic Catalan objects of order n:

$$1, 1, 3, 12, 57, 301, 1707, 10191, 63244, 404503, \ldots \text{(OEIS A151498)}$$

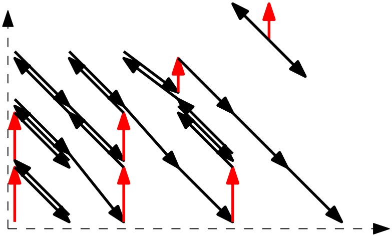
$=$ certain walks in the quadrant

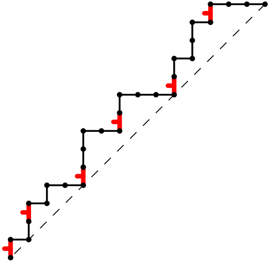
- Bijective link? An easier-to-understand structure?

Marked paths and steep pairs

Walks in the quadrant: $\{(1,0),(1,-1),(-1,1)\}$, ending with $y=0$.

Considered in (Bousque-Mélou and Mishna, 2010) and counted in (Mishna and Rechnitzer, 2009)

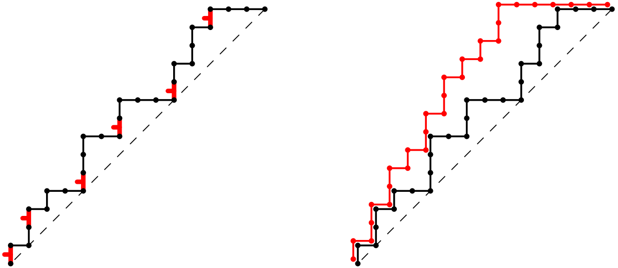




In bijection with level-marked Dyck paths:
level $\leq$ marking before the point

Level-marked Dyck paths and steep pairs

Steep pairs : 2 nested Dyck paths, the one above has no EE except at the end



Bijection:

- Path below: path without marking
- Path above: read the N's, marked $\to N$, not marked $\to EN$

Steep-Bounce conjecture

Conjecture (Bergeron, Ceballos and Pilaud 2018+, Conjecture 2.2.8)

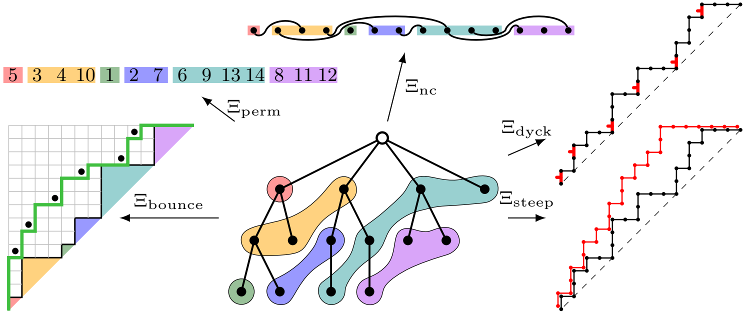
The following two sets are of the same size:

- *bounce pairs of order n with k blocks;*
- *steep pairs of order n with k east steps E on $y = n$.*

A proof gives the counting of all these objects (pipe dreams and parabolic Catalan)

The cases $k = 1, 2, n-1, n$ already proved

A scheme of the bijections



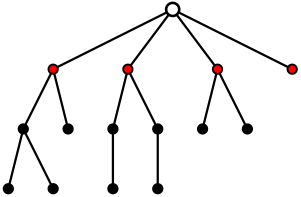
Left-aligned colored trees

- T : plane tree with n non-root nodes;
- $\alpha = (\alpha_1, \ldots, \alpha_k)$: composition of n

Active nodes : not yet colored, but parent is colored or is the root.

Coloring algorithm : For i from 1 to k,

- If there are less than α_i active nodes, then fail;
- Otherwise, color the first α_i from left to right with color i.



$\alpha = (\mathbf{1}, 3, 1, 2, 4, 3) \vdash 14$

When succeeded, it is a left-aligned colored tree (or a LAC tree).

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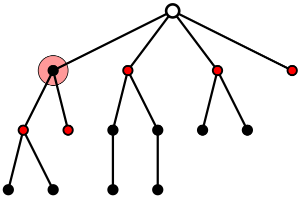

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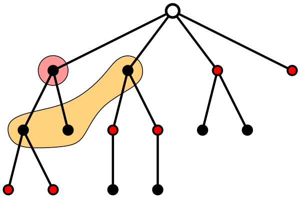

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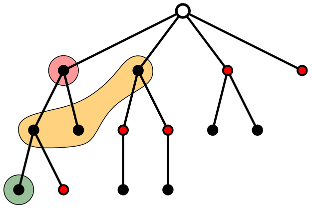

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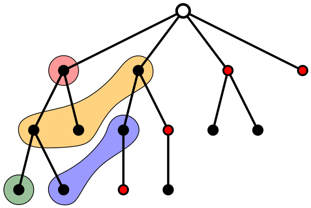

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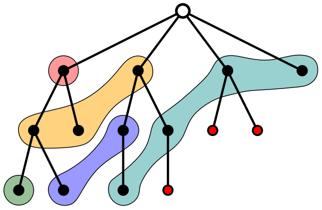

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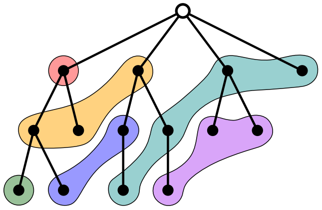

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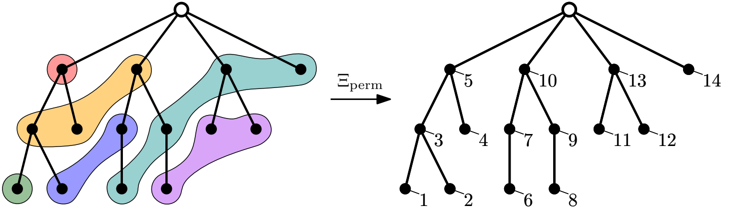
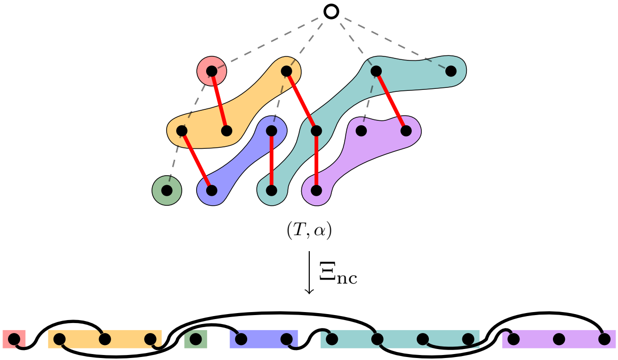
To permutations



(T, α)

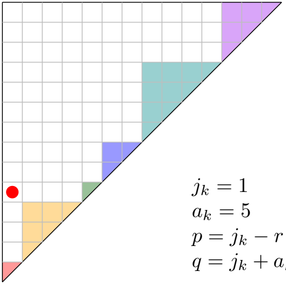
$\Xi_{\text{perm}}(T, \alpha) = 5 \mid 3\ 4\ 10 \mid 1 \mid 2\ 7 \mid 6\ 9\ 13\ 14 \mid 8\ 11\ 12 \in \mathfrak{S}_n^{\alpha}(231)$

To parabolic non-crossing partitions



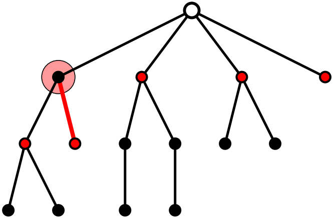
- LAC tree → partition : flatten the layers
- Partition → LAC tree : look at the sky

To bounce pairs



$j_k = 1$
$a_k = 5$
$p = j_k - r + 1 = 1$
$q = j_k + a_k - s = 4$

$r = 1$
$s = 2$



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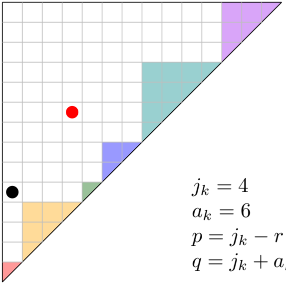
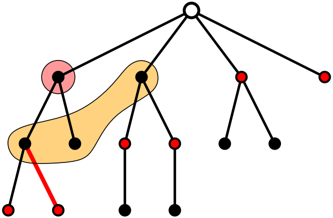


$j_k = 4$
$a_k = 6$
$p = j_k - r + 1 = 4$
$q = j_k + a_k - s = 8$

$r = 1$
$s = 2$



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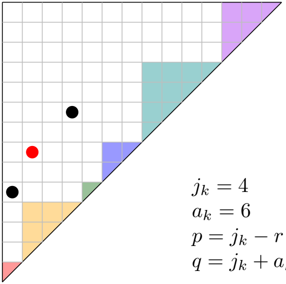


$j_k = 4$

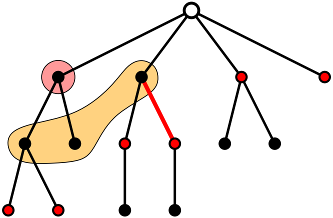
$a_k = 6$

$p = j_k - r + 1 = 2$

$q = j_k + a_k - s = 6$

$r = 3$

$s = 4$



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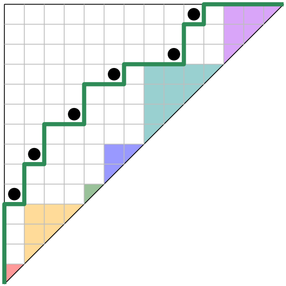
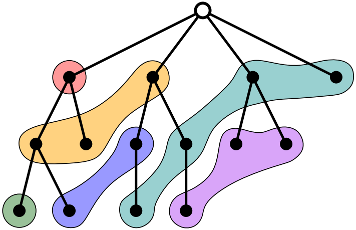
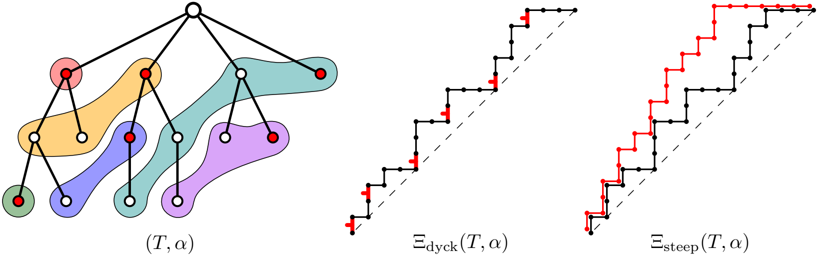




$\alpha = (1,3,1,2,4,3) \vdash 14$

To steep pairs



Steep-Bounce theorem

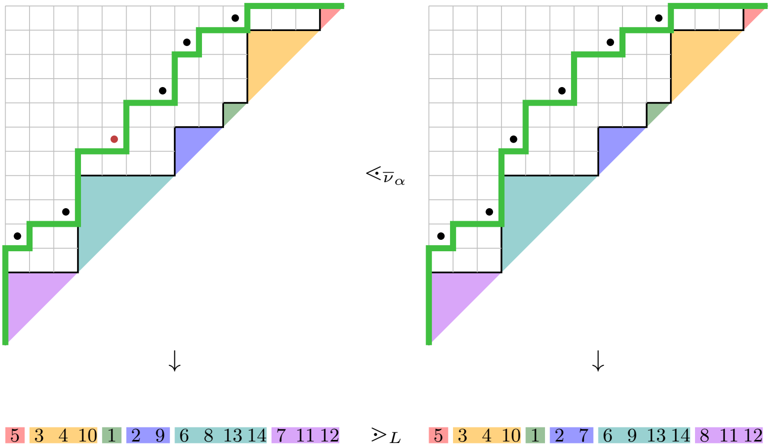
Theorem (Ceballos, F., Mühle 2018+)

There is a natural bijection Γ between the following two sets:

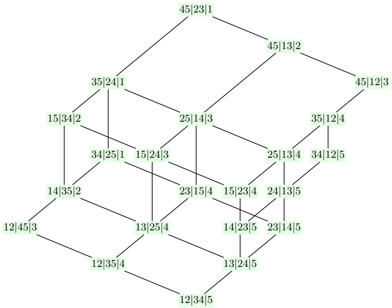
- *bounce pairs of order n with k blocks;*
- *steep pairs of order n with k each steps E on $y = n$.*

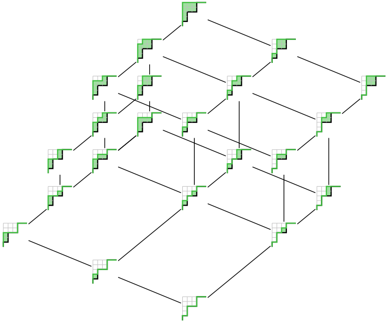
So we know how to count them!

A bijection between the two Tamaris



One isomorphic to the dual of the other

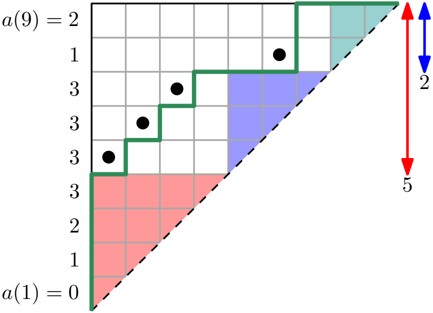




Theorem (Ceballos, F., Mühle 2018+)

The parabolic Tamari lattice indexed by α is isomorphic to the ν-Tamari lattice with $\nu = N^{\alpha_1}E^{\alpha_1}\cdots N^{\alpha_k}E^{\alpha_k}$.

Detour to q, t-Catalan combinatorics



$\text{area}(D) = \sum_i a(i) = 18$

$\text{bounce}(D) = \sum_i (i-1)\alpha_i = 7$

$\text{dinv}(D) = \#\{(i,j) \mid i < j, (a(i) = a(j) \vee a(i) = a(j)+1\} = 17$

A non-trivial symmetry

Theorem (Garsia and Haiman 1996, Haiman 2001)

By summing up all Dyck paths of order n, we have

$$\sum_D q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)} = \sum_D q^{\mathrm{bounce}(D)} t^{\mathrm{area}(D)}.$$

The proof goes by the Hilbert series of the diagonal coinvariant space with two sets of variables.

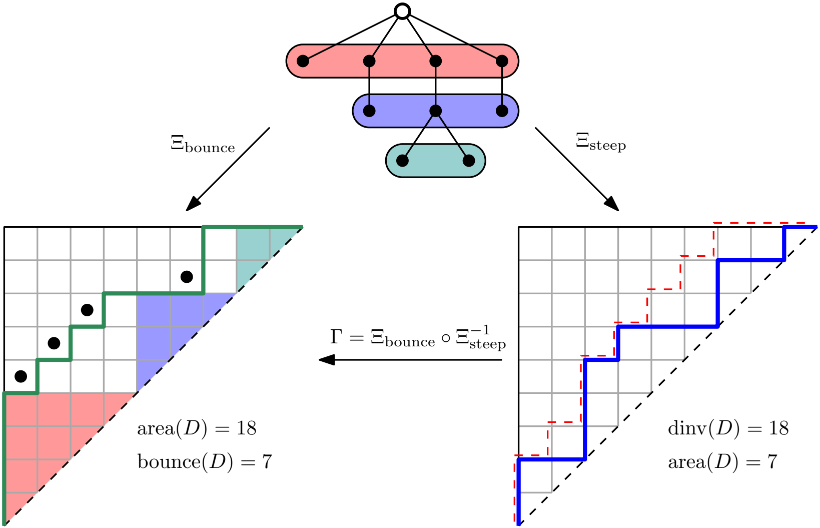
No combinatorial proof!

Theorem (Haglund 2008, Proof of Theorem 3.15)

There is a bijection ζ on Dyck paths that transfers the pairs of statistics

$$(\mathrm{dinv}, \mathrm{area}) \to (\mathrm{area}, \mathrm{bounce}).$$

Our zeta map


Ξ_{bounce}
Ξ_{steep}
$\Gamma = \Xi_{\text{bounce}} \circ \Xi_{\text{steep}}^{-1}$
$\text{area}(D) = 18$
$\text{bounce}(D) = 7$
$\text{dinv}(D) = 18$
$\text{area}(D) = 7$

Our zeta map, Steep-Bounce version

Theorem (Ceballos, F., Mühle 2018+)

There is a natural bijection Γ between the following sets:

- *bounce pairs of order n with k blocks;*
- *steep pairs of order n with k east steps E on $y = n$.*

ζ = special case of Γ, with steep pairs and bounce pairs constructed in a greedy way

A generalization to explore!

Possible directions

- Many questions in enumeration (but possibly very difficult)
- How are the statistics transferred, and which ones?
- Action by symmetries?
- Implication in diagonal coinvariant spaces?
- *etc.* ?

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Thank you for listening!