Bijections between planar maps and planar linear normal λ-terms with connectivity condition

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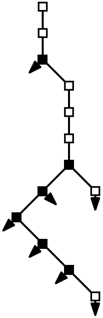
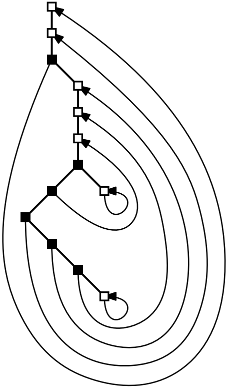
λ-terms, according to a combinatorialist

λ-term: unary-binary tree (skeleton) + variable-abstraction map

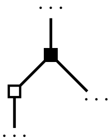
Linear λ-term: the variable-abstraction map being bijective

$$t = \lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.(v(w(x(\lambda y.y))))z(\lambda k.k))$$





Some special properties

- closed: the variable-abstraction map is complete
- unitless: no closed sub-term
- normal: no β-reduction, *i.e.*, avoiding 
- (RL-)planar: right-to-left variable-abstraction map
  Example: $t = \lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.(v(w(x(\lambda y.y))))z(\lambda k.k))$

All can be translated combinatorially to trees!

Linear + planar : unique choice, so just unary-binary tree!

Known enumerations

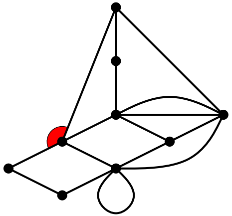
λ-terms	Maps	OEIS
linear	general cubic	A062980
planar	planar cubic	A002005
unitless	bridgeless cubic	A267827
unitless planar	bridgeless planar cubic	A000309
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	bridgeless planar	A000260

Noam Zeilberger, *A theory of linear typings as flows on 3-valent graphs*, LICS 2018

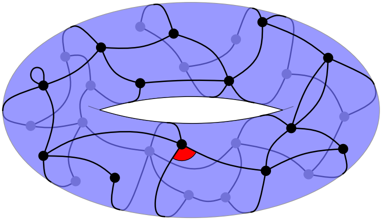
A lot of people and work: Zeilberger, Bodini, Gardy, Jacquot, Giorgetti, Courtiel, Yeats, ...

What is a map?

Combinatorial map: drawing of graphs on a surface



sphere/planar ($g = 0$)



torus ($g = 1$)

We only consider rooted map, *i.e.*, with a marked corner.

Different families of maps

- Planar: maps on the plane
- Cubic: all vertices have degree 3
- Bridgeless: remain connected after removal of any one edge
- Loopless: no loop (edge linking a vertex with itself)
- Bipartite: has a proper 2-coloring on vertices
- ...

Some are related by duality, which turns vertices into faces and *vice versa*.

Connectivity condition

Noam Zeilberger and Jason Reed (CLA 2019)

How about connectivity of the diagram on **planar linear normal terms**?

k-connected: breaking $k-1$ edges does not split the graph

- 1-connected: all (connected by their skeleton)
- 2-connected: unitless (bridge $\Leftrightarrow$ closed sub-term)
- 3-connected: ???

Conjecture (Zeilberger–Reed, 2019)

The number of 3-connected planar linear normal λ-terms with $n+2$ variables is

$$\frac{2^n}{(n+1)(n+2)}\binom{2n+1}{n},$$

which also counts bipartite planar maps with n edges (A000257).

Also claimed characterization with typing for k-connected terms.

Our contribution (1)

Katarzyna Grygiel and Guan-Ru Yu (CLA 2020): combinatorial characterization of 3-connected terms, partial bijective results

Theorem (Fang, 2022+)

There is a direct bijection between 3-connected planar linear normal λ-terms with $n+2$ variables and bipartite planar maps with n edges.

Bijection $\Rightarrow$ transfer of statistics $\Rightarrow$ generating function $\Rightarrow$ asymptotics behavior

Our contribution (2)

Theorem (Fang, 2022+)

There is a direct bijection from planar linear normal λ-terms to planar maps. Furthermore, when restricted to unitless terms, the bijection leads to loopless planar maps.

λ-terms	Maps	OEIS
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	loopless planar	A000260

Known recursive bijection in (Zeilberger and Giorgetti, 2015) via LR-planar terms

Done with a new recursive decomposition of planar maps

From λ-terms to unary-binary trees

Linear planar λ-terms $\Leftrightarrow$ unary-binary trees (with conditions)

Three statistics for a unary-binary tree S:

- $\mathrm{unary}(S)$: number of unary nodes (abstractions)
- $\mathrm{leaf}(S)$: number of leaves (variables)
- $\mathrm{excess}(S)$: $\mathrm{leaf}(S) - \mathrm{unary}(S)$ (free variables)

Properties of linear planar λ-terms $\Leftrightarrow$ properties on skeletons

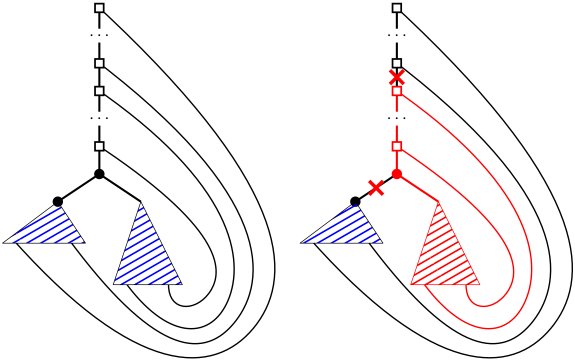
S_u: sub-tree of S induced by u

- Linear $\Leftrightarrow$ $\mathrm{excess}(S) = 0$
- Normal $\Leftrightarrow$ Left child of a binary node is never unary
- 1-connected $\Leftrightarrow$ $\mathrm{excess}(S_u) \geq 0$ for all u
- 2-connected $\Leftrightarrow$ $\mathrm{excess}(S_u) > 0$ for all u non-root

Characterization of 3-connectedness (1)

Proposition (Grygiel and Yu, CLA 2020)

Let S be the skeleton of a 3-connected planar linear λ-term, then the left child of the first binary node is a leaf.



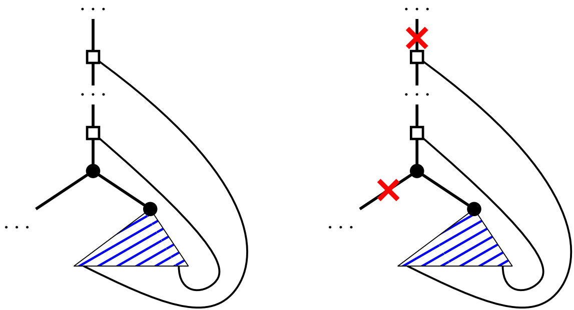
Reduced skeleton: the right sub-tree of the first binary node

Characterization of 3-connectedness (2)

Proposition (Proposed by Grygiel and Yu, CLA 2020)

S *is the reduced skeleton of a 3-connected planar linear normal* λ*-term iff*

- **(Normality)** *The left child of a binary node in* S *is never unary;*
- **(3-connectedness)** *For every binary node* u *with* v *its right child,* $\mathrm{excess}(S_v)$ *is strictly larger than the number of consecutive unary nodes above* u*.*



Clearly necessary, but also sufficient!

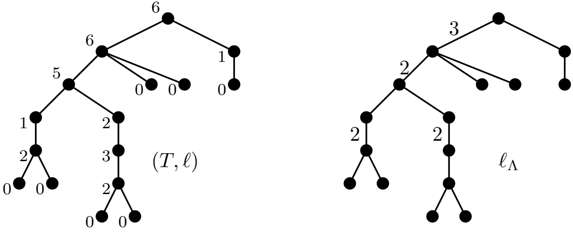
Degree trees

Degree tree: a plane tree T with a labeling ℓ on nodes with

- u is a leaf $\Rightarrow \ell(u) = 0$;
- u has children $v_1, \dots v_k \Rightarrow s(u) - \ell(v_1) \le \ell(u) \le s(u)$, where $s(u) = k + \sum_{i=1}^{k} \ell(v_i)$.

Contribution of each child : 1 (itself) + $\ell(v_i)$ (its sub-tree)

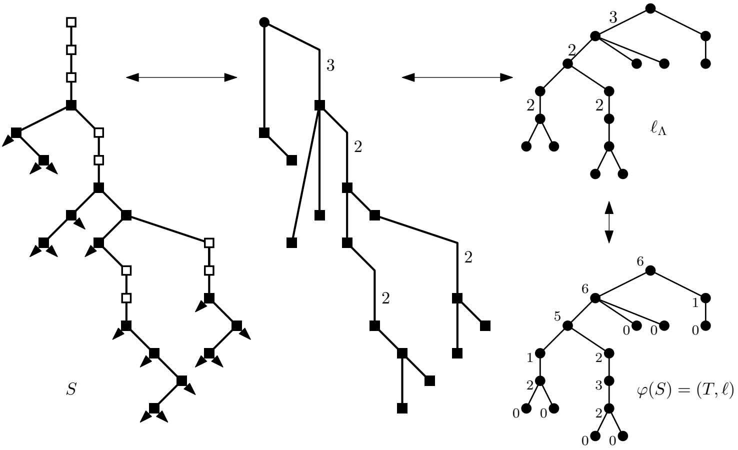
Except for the first child: from 1 to its due contribution.



Edge labeling ℓ_Λ: the subtracted contribution

ℓ and ℓ_Λ interchangeable!

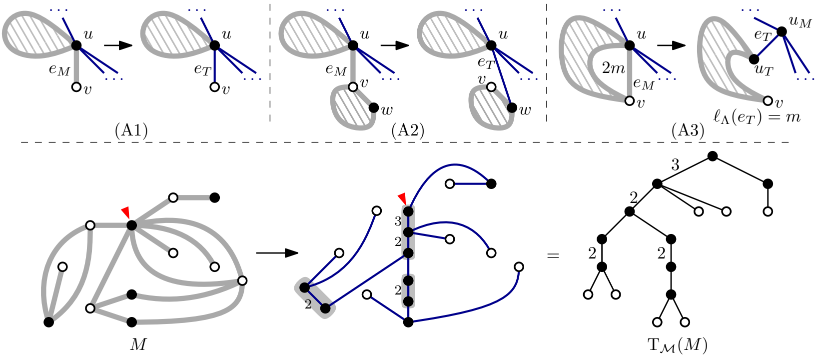
First bijection (1/2): 3-connected terms $\Leftrightarrow$ degree trees



- Unary nodes on right child $\Leftrightarrow$ Subtraction on left child
- Leftmost leaf $\Leftrightarrow$ Contribution 1

Related to Böhm trees.

First bijection (2/2): degree trees ⇔ bipartite planar maps



Existing direct bijection (F., 2021), using an exploration

Also related to Chapoton's new intervals in the Tamari lattice

Some statistics correspondences:

- Unary chains of length k ⇔ edge label k ⇔ faces of degree $2k$
- Initial unary chain ⇔ root label ⇔ degree of root face

Consequence in enumeration

Bijections are useful!

- Transfer of statistics (also about applications in λ-terms)
- Generating function for free!
- Probabilistic results also for free!

Proposition (F. 2022+)

Let $X_n = \#$ initial abstractions of a uniformly random 3-connected planar linear normal λ-term. When $n \to \infty$,

$$\mathbb{P}[X_n = k] \to \frac{k-1}{3}\binom{2k-2}{k-1}\left(\frac{3}{16}\right)^{k-1}.$$

Corollary of known results on bipartite maps

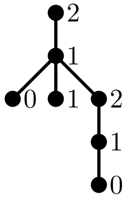
Connected terms and trees

Recall the conditions:

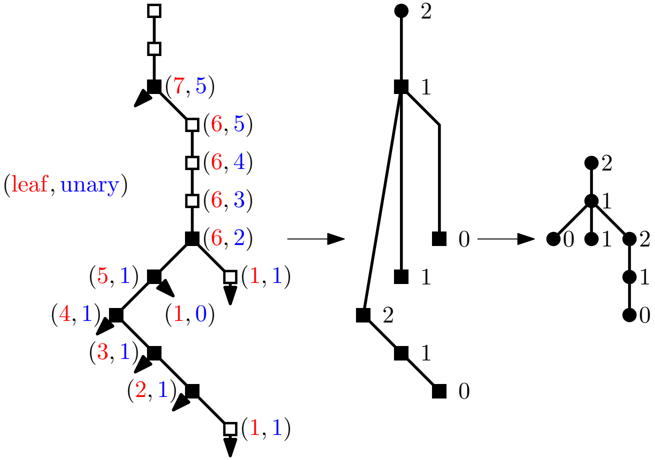
- Linear $\Leftrightarrow \text{excess}(S) = 0$
- Normal $\Leftrightarrow$ Left child of a binary node is never unary
- 1-connected $\Leftrightarrow \text{excess}(S_u) \geq 0$ for all u
- 2-connected $\Leftrightarrow \text{excess}(S_u) > 0$ for all u non-root

v-trees: a plane tree T with a labeling ℓ on nodes with

- Leaves $u \Rightarrow \ell(u) \in \{0, 1\}$;
- Non-root u with children $v_1, \ldots, v_k \Rightarrow 0 \leq \ell(u) \leq 1 + \sum_{i=1}^{k} \ell(v_i)$



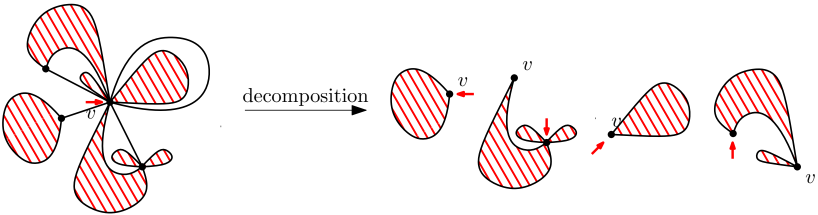
Second bijection (1/2)



$\lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.(v(w(x(\lambda y.y))))z(\lambda k.k))$

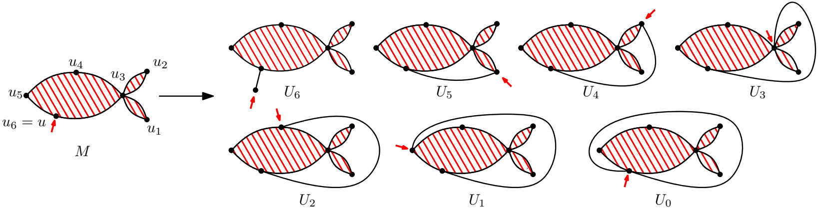
Excess of the right child! $0 \Leftrightarrow$ closed sub-term

One-corner decomposition of maps



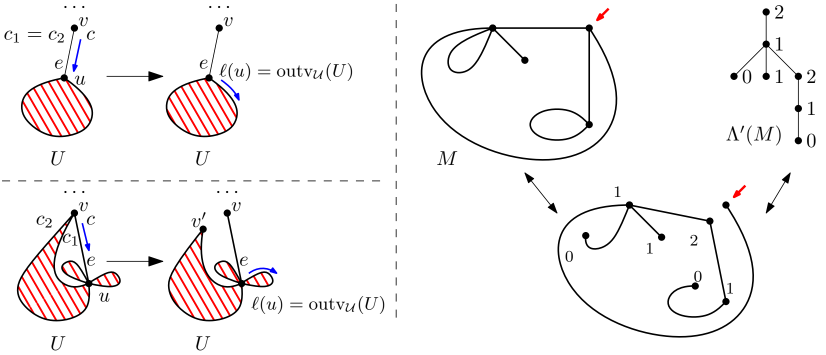
One-corner component: maps whose root vertex has one outer corner

Catalytic statistics: number of non-root vertices on outer face ($\mathrm{outv}_{\mathcal{U}}$)



Second bijection (2/2)

Direct bijection = "de-recusifyng" the decomposition



Loop ⇔ one-corner component with single outer node ⇔ label 0 in tree

Natural specialization to loopless planar maps ⇔ unitless terms!

Recapitulation

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β-normal unitless planar	loopless planar	A000260
β-normal 3-connected planar	bipartite planar	A000257

First bijection (direct) for 3-connected planar normal terms and bipartite planar maps

New bijection (direct) for general planar normal terms and planar maps, naturally restricted to 2-connected terms and loopless planar maps

Not the same bijection... But in the same spirit.

Higher connectivity? Types? Other enumeration consequences?

Recapitulation

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β-normal linear/$\sim$	general	A000698
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Thank you for listening!