Fighting fish and two-stack sortable permutations

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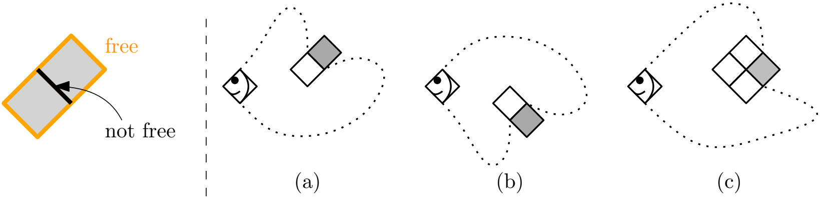
Fighting fish

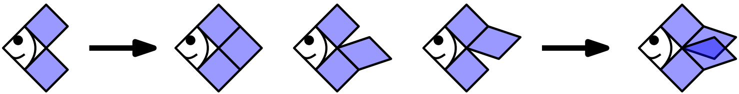
A **fighting fish** = gluing of unit cells, generalizing directed polyominoes

- either a single cell (the head);
- or obtained from gluing a cell to a fighting fish as follows.

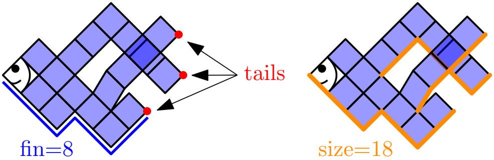


Gluing only to (upper or lower) right free edges!

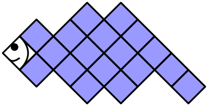
Order of gluing does not matter, and it is not a 2D object!



Anatomy of fighting fish



- Area = # cells
- Fin = length of path via lower free edges to first tail
- Size = # lower free edges

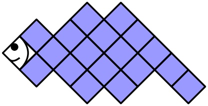


Fighting fish with one tail = parallelogram polyominoes

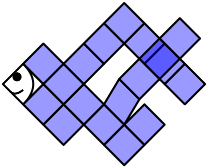
Size = Semi-perimeter

Why fighting fish?

Parallelogram polynomioes of size $n \Rightarrow$ average area $\Theta(n^{3/2})$



Duchi, Guerrini, Rinaldi and Schaeffer 2016:
Fighting fish of size $n \Rightarrow$ average area $\Theta(n^{5/4})$



A new and interesting model of branching surfaces!

Enumeration of fighting fish

Fighting fish with one tail (parallelogram polynominoes) of size $n+1$:

$$\mathrm{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}.$$

Duchi, Guerrini, Rinaldi and Schaeffer 2016:

Fighting fish of size $n+1$:

$$\frac{2}{(n+1)(2n+1)}\binom{3n}{n}.$$

The same formula applies to

- non-separable planar maps;
- two-stack sortable permutations;
- left ternary trees;
- generalized Tamari intervals;
- etc...

Enumeration of fighting fish, refined

Duchi, Guerrini, Rinaldi and Schaeffer 2017:

Fighting fish of size $n+1$, with i lower-left free edges and j lower-right free edges ($i+j=n+1$):

$$\frac{1}{(2i+j-1)(2j+i-1)}\binom{2i+j-1}{i}\binom{2j+i-1}{j}.$$

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Also the number of non-separable planar maps with n edges, $i+1$ vertices and $j+1$ faces (*cf.* Brown and Tutte 1964);

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Also the number of two-stack sortable permutations of length n, with i ascents and j descents (*cf.* Goulden and West 1996);

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Also the number of two-stack sortable permutations of length n, with i ascents and j descents (*cf.* Goulden and West 1996);

Also the number of left ternary trees with i even vertices and j odd vertices (*cf.* Del Lungo, Del Ristoro and Penaud 1999) ...

A conjecture for a bijection

Conjecture (Duchi, Guerrini, Rinaldi and Schaeffer 2016)

The number of fighting fish with

- n *as size,*
- k *as fin length,*
- ℓ *tails,*
- i *left-lower free edge, and*
- j *right-lower free edge*

is equal to the number of left ternary trees with

- n *nodes,*
- k *as core size,*
- ℓ *right branches,*
- $i+1$ *non-root nodes with even abscissa, and*
- j *nodes with odd abscissa.*

So refined, we may as well ask for a bijection!

Our result

Theorem (F. 2018+)

There is a bijection between fighting fish with

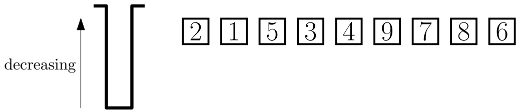
- n *as size,*
- k *as fin length,*
- ℓ *tails,*
- i *left-lower free edge, and*
- j *right-lower free edge*

and two-stack sortable permutations with

- $n-1$ *elements,*
- $k-1$ *left-to-right maxima in the permutation sorted once,*
- $\ell-1$ *left descents in the permutation sorted once,*
- $i-1$ *ascents, and*
- $j-1$ *descents.*

Not exactly the conjecture, but in its spirit.

Sorting a permutation with a stack



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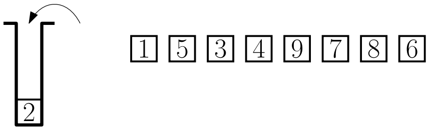

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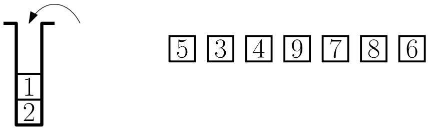

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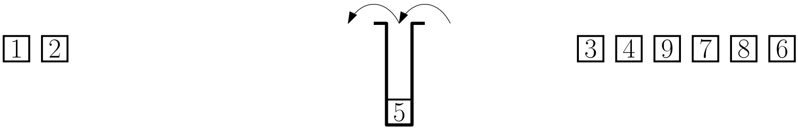

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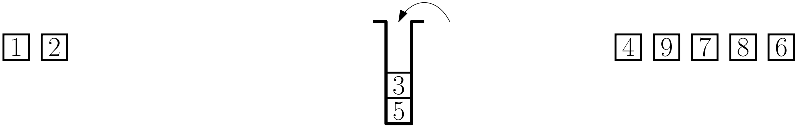

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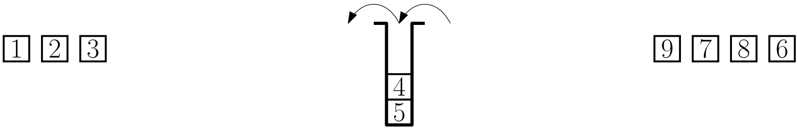

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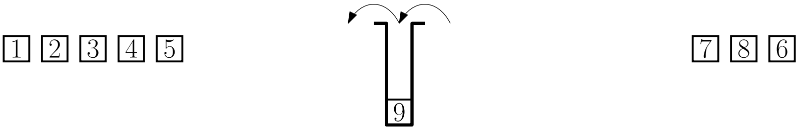

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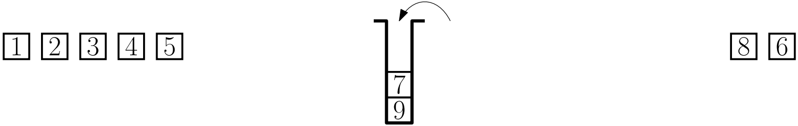

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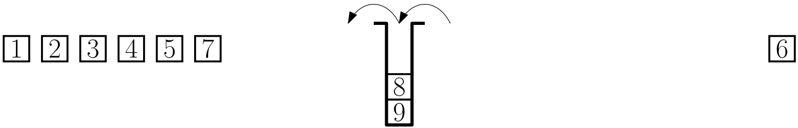

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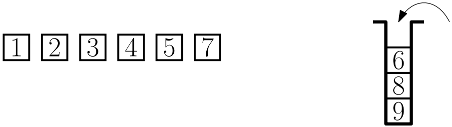

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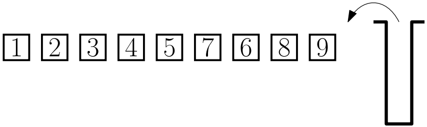
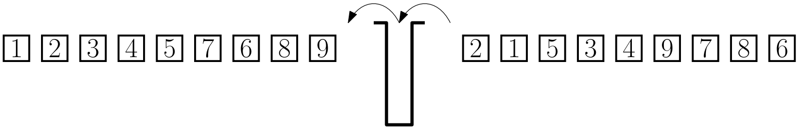


Stack-sortable permutations

A permutation is **stack-sortable** if it is sorted in one pass.



Examples: 21534 is stack-sortable, but 215349786 is not

Theorem (Knuth 1968)

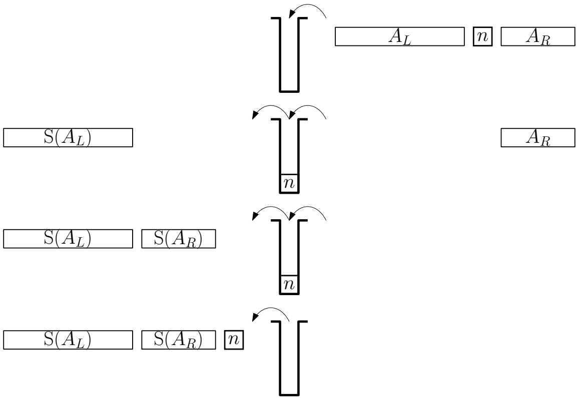
A permutation is stack-sortable iff it contains no pattern 231.

The number of stack-sortable permutations of length n is the n-th Catalan number $\mathrm{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}$.

What about two passes?

Sorting operator

S: operator of stack sorting (valid for general sequences)



$$\mathrm{S}(\epsilon) = \epsilon, \quad \mathrm{S}(A_L \cdot n \cdot A_R) = \mathrm{S}(A_L) \cdot \mathrm{S}(A_R) \cdot n.$$

Two-stack sortable permutations

A permutation $\pi \in \mathfrak{S}_n$ is

- a stack-sortable permutation if $\mathrm{S}(\pi) = 12\ldots n$
- a **two-stack sortable permutation** (or 2SSP) if $\mathrm{S}(\mathrm{S}(\pi)) = 12\ldots n$.

Theorem (West 1991, Zeilberger 1992)

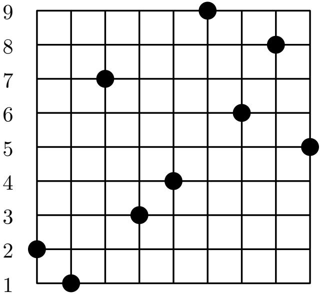
The number of 2SSPs of length n is

$$\frac{2}{(n+1)(3n+1)}\binom{3n+1}{n}.$$

Also characterization with forbidden pattern.

We will now look at a new recursive decomposition.

Permutation on a grid



$\pi = 2\ 1\ 7\ 3\ 4\ 9\ 6\ 8\ 5$

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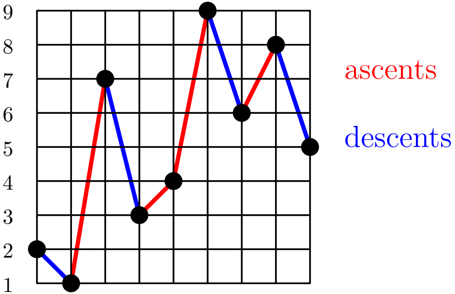

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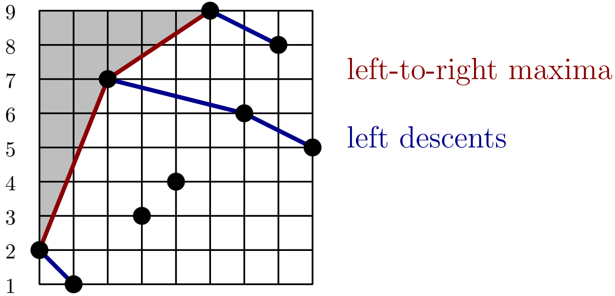

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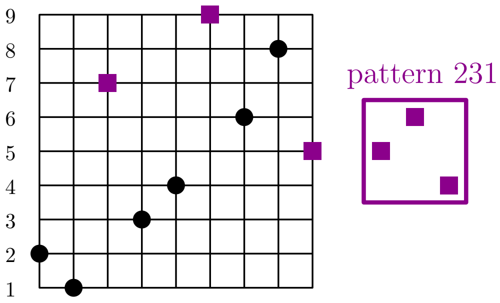

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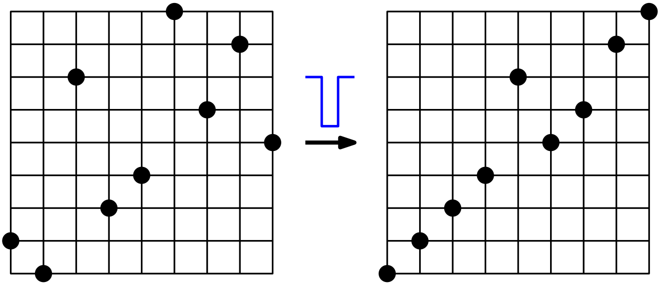
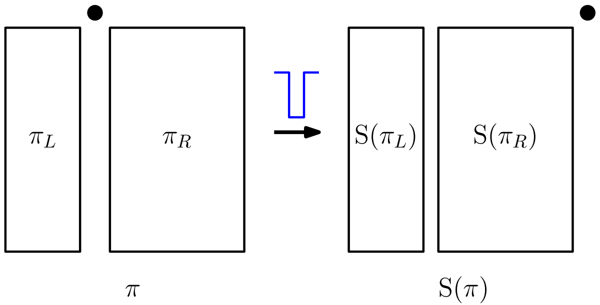
A characterization



$\pi = 2\ 1\ 7\ 3\ 4\ 9\ 6\ 8\ 5$ $\qquad$ $\mathrm{S}(\pi) = 2\ 1\ 7\ 3\ 4\ 9\ 6\ 8\ 5$

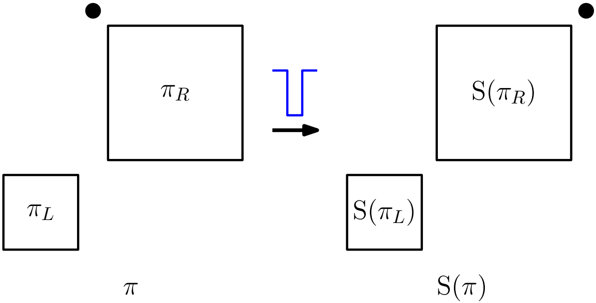
π is two-stack sortable $\Leftrightarrow$ $\mathrm{S}(\pi)$ avoids pattern 231

Decomposing...



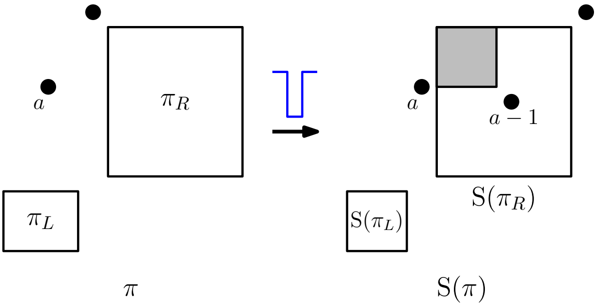
We recall that $S(\pi_L \cdot n \cdot \pi_R) = S(\pi_L) \cdot S(\pi_R) \cdot n$.
When compactified, both π_L and π_R are two-stack sortable.

Case 1



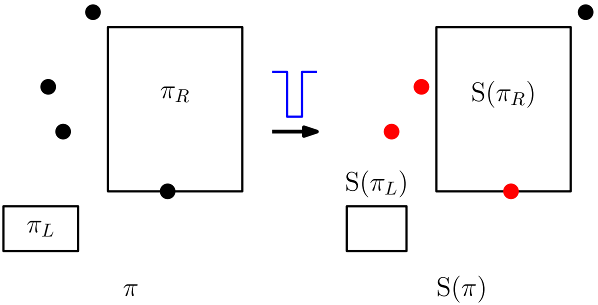
When every element of π_L are smaller than the min of π_R, it is easy. Just put them side by side. π_L and π_R can be empty.

Case 2



When only one element a of π_L is larger than the min of π_R, then $a-1$ is a left-to-right maximal in $S(\pi_R)$. π_L and π_R cannot be empty.

Case 3 ... ?



It is impossible to have two elements of π_L larger than the min of π_R, if we want to avoid 231 in $\mathrm{S}(\pi)$.

Recursive construction

$\mathrm{slmax}(\pi) = \#$ left-to-right maxima in $\mathrm{S}(\pi)$

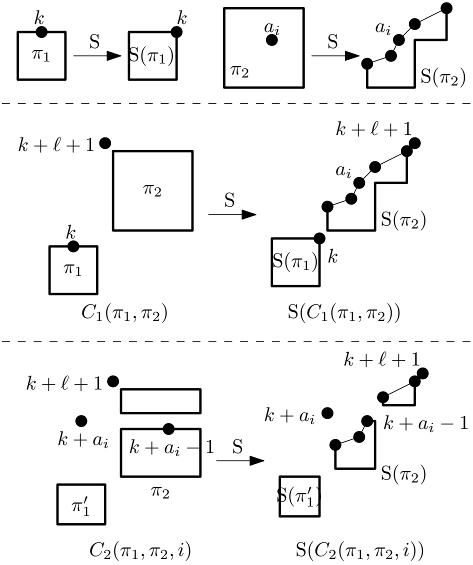
For π_1, π_2 2SSPs, we get

- $C_1(\pi_1, \pi_2)$
- $C_2(\pi_1, \pi_2, i)$ for $1 \le i \le \mathrm{slmax}(\pi_2)$

Here,

$\mathrm{slmax}(C_1(\pi_1, \pi_2)) = \mathrm{slmax}(\pi_1) + \mathrm{slmax}(\pi_2) + 1,$

$\mathrm{slmax}(C_2(\pi_1, \pi_2, i)) = \mathrm{slmax}(\pi_1) + \mathrm{slmax}(\pi_2) - i + 1.$



Various statistics

Proposition

Given two 2SSPs π_1, π_2, *for any* i *with* $1 \leq i \leq \mathrm{slmax}(\pi_2)$, *we have*

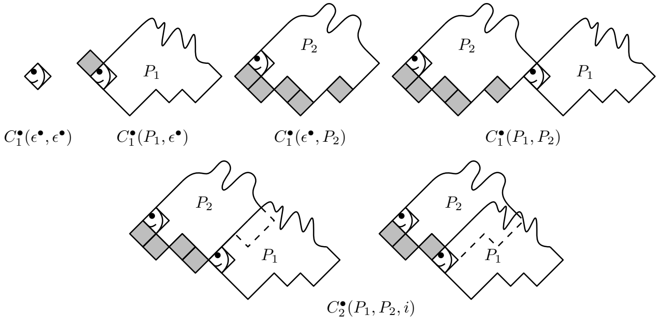
$$\mathrm{asc}(C_1(\pi_1,\pi_2)) = \mathrm{asc}(C_2(\pi_1,\pi_2,i)) = \mathrm{asc}(\pi_1) + 1 + \mathrm{asc}(\pi_2),$$
$$\mathrm{des}(C_1(\pi_1,\pi_2)) = \mathrm{des}(C_2(\pi_1,\pi_2,i)) = \mathrm{des}(\pi_1) + 1 + \mathrm{des}(\pi_2),$$
$$\mathrm{len}(C_1(\pi_1,\pi_2)) = \mathrm{len}(C_2(\pi_1,\pi_2,i)) = \mathrm{len}(\pi_1) + 1 + \mathrm{len}(\pi_2),$$
$$\mathrm{sldes}(C_1(\pi_1,\pi_2)) = \mathrm{sldes}(\pi_1) + \mathrm{sldes}(\pi_2),$$
$$\mathrm{sldes}(C_2(\pi_1,\pi_2,i)) = \mathrm{sldes}(\pi_1) + \mathrm{sldes}(\pi_2) + 1.$$

When one of π_1, π_2 *is empty, the formulas for* $C_1(\pi_1,\pi_2)$ *still hold, except that* $\mathrm{asc}(C_1(\epsilon,\pi_2)) = \mathrm{asc}(\pi_2)$ *and* $\mathrm{des}(C_1(\pi_1,\epsilon)) = \mathrm{des}(\pi_1)$.

Wasp-waist decomposition of fighting fish

Duchi, Guerrini, Rinaldi and Schaeffer 2017:

Idea: delete cells on lower left one by one, until it breaks ($\text{fin}(\epsilon^\bullet) = 1$)



$$\text{fin}(C_1^\bullet(P_1, P_2)) = \text{fin}(P_1) + \text{fin}(P_2)$$
$$\text{fin}(C_2^\bullet(P_1, P_2, i)) = \text{fin}(P_1) + \text{fin}(P_2) - i \quad (1 \le i \le \text{fin}(P_2) - 1)$$

Isomorphic decompositions with 2SSPs!

Statistics also agree!

Convention: $\mathrm{lsize}(\epsilon^\bullet) = \mathrm{rsize}(\epsilon^\bullet) = \mathrm{size}(\epsilon^\bullet) = \mathrm{tails}(\epsilon^\bullet) = 1$

Proposition (Duchi, Guerrini, Rinaldi and Schaeffer 2017)

Given two fighting fish P_1, P_2, for i with $1 \leq i \leq \mathrm{fin}(P_2) - 1$, we have

$$\begin{aligned}
\mathrm{lsize}(C_1^\bullet(P_1, P_2)) = \mathrm{lsize}(C_2^\bullet(P_1, P_2, i)) &= \mathrm{lsize}(P_1) + \mathrm{lsize}(P_2)\\
\mathrm{rsize}(C_1^\bullet(P_1, P_2)) = \mathrm{rsize}(C_2^\bullet(P_1, P_2, i)) &= \mathrm{rsize}(P_1) + \mathrm{rsize}(P_2)\\
\mathrm{size}(C_1^\bullet(P_1, P_2)) = \mathrm{size}(C_2^\bullet(P_1, P_2, i)) &= \mathrm{size}(P_1) + \mathrm{size}(P_2)\\
\mathrm{tails}(C_1^\bullet(P_1, P_2)) &= \mathrm{tails}(P_1) - 1 + \mathrm{tails}(P_2)\\
\mathrm{tails}(C_2^\bullet(P_1, P_2, i)) &= \mathrm{tails}(P_1) + \mathrm{tails}(P_2)
\end{aligned}$$

The formulas for $C_1^\bullet(P_1, P_2)$ hold for P_1 or P_2 being $\epsilon^\bullet$, except that $\mathrm{lsize}(C_1^\bullet(\epsilon^\bullet, P_2)) = \mathrm{lsize}(P_2)$, and $\mathrm{rsize}(C_1^\bullet(P_1, \epsilon^\bullet)) = \mathrm{rsize}(P_1)$.

Bijection

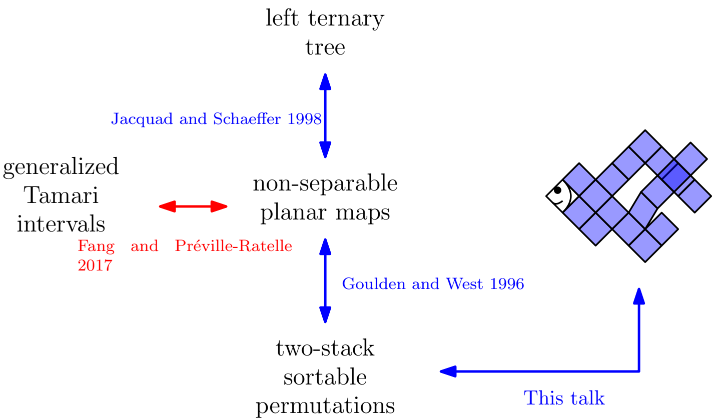
Isomorphic recursive decompositions of fighting fish and two-stack sortable permutations, with many agreeing statistics

$$\Rightarrow$$

Recursive bijection preserving the statistics

Also possible to write functional equations, and we prove that the generating function with all these statistics is algebraic.

Direct bijection?



Any direct bijection?

Open problems

- Symmetries?
- Some statistic corresponding to area?
- How about sorting three (four, five, ...) times through a stack?

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- How about sorting three (four, five, ...) times through a stack?

Thank you for your attention!