Tamari intervals and blossoming trees

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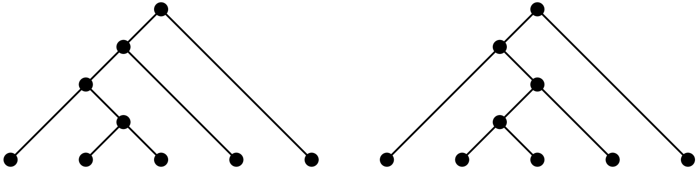
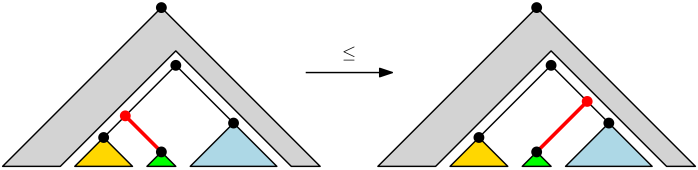
Binary trees

Binary trees : n binary internal nodes and $n+1$ leaves



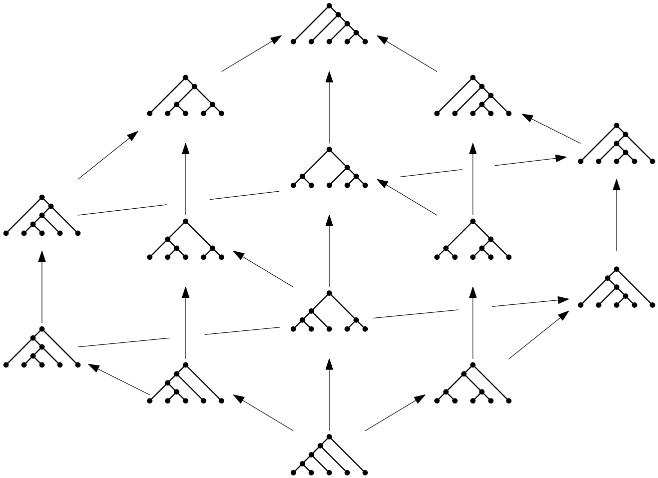
Counted by Catalan numbers: $\mathrm{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}$

Rotation (from left to right) :



Tamari lattice

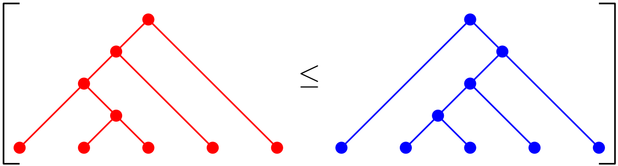
Left-to-right rotation defines a self-dual lattice (Tamari 1962)



Deep links with subjects in combinatorics, and many generalizations!

The next level: intervals

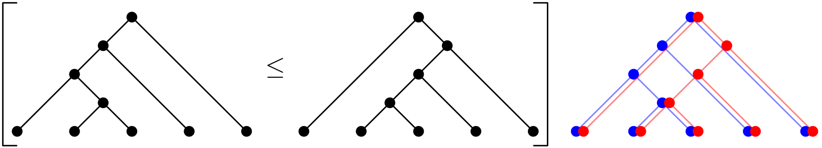
A Tamari interval: $[S, T]$ of binary trees with $S \leq T$



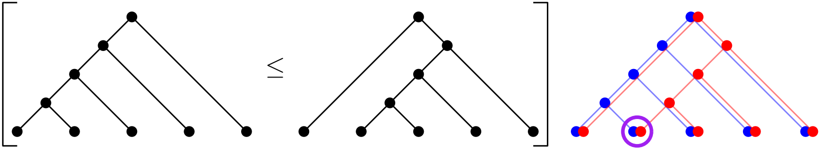
Motivation: conjecturally related to trivariate diagonal coinvariant spaces, also with operads... and nice numbers!

Many different families

Synchronized intervals: leaves on the same direction



synchronized

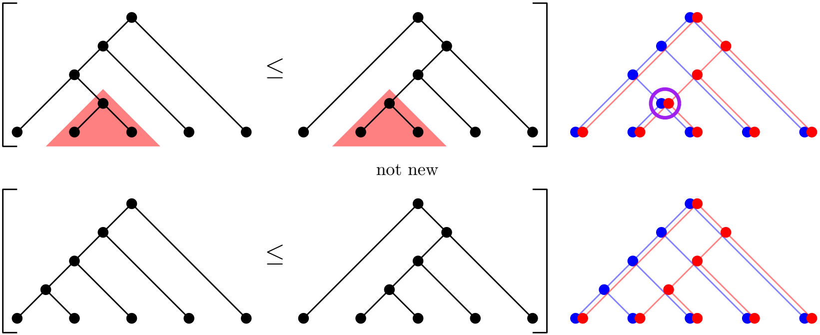


not synchronized

In bijection with ν-Tamari intervals (Préville-Ratelle–Viennot 2017)

Many different families (cont.)

New/modern intervals (Chapoton): no shared internal nodes



not new

new

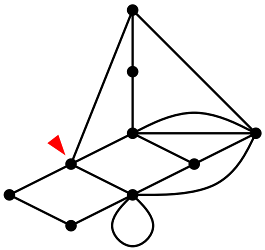
First defined for enumeration, with algebraic and geometric links

- Infinitely modern intervals: further restriction
- Kreweras intervals: algebraic link

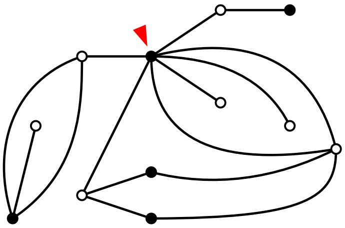
They are often in bijection with families of planar maps!

What is a planar map?

Planar map: drawing of graphs on a plane without extra crossing



planar



bipartite planar

They are rooted, *i.e.*, with a marked corner.

Also many interesting families: triangulation, bipartite, ...

Tamari intervals and planar maps

Intervals	Formula	Planar maps
General	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	bridgeless 3-connected triangulation
Synchronized	$\frac{2}{n(n+1)}\binom{3n}{n-1}$	non-separable
New/modern	$\frac{3\cdot 2^{n-2}}{n(n+1)}\binom{2n-2}{n-1}$	bipartite
Kreweras	$\frac{1}{2n+1}\binom{3n}{n}$	stacked triangulation

Also in bijection with other objects: interval posets, closed flow in forest, fighting fish, λ-term, ...

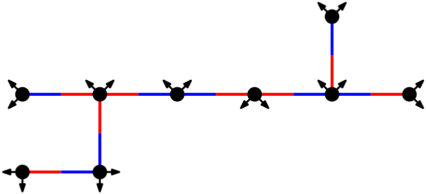
Many have worked on them: Bernardi, Bonichon, Bousquet-Mélou, Ceballos, Chapoton, Châtel, Chenevière, Combe, Duchi, F., Fusy, Henriet, Humbert, Préville-Ratelle, Pons, Rognerud, Viennot, Zeilberger, ...

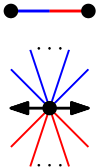
But a different equation / bijection for each family...

Our results

(Bicolored) Blossoming tree: an unrooted plane tree such that

- Each edge is half red and half blue.
- Each node has two **buds**, splitting reds and blues.





Many variants, used a lot in enumeration of maps (Poulalhon–Schaeffer 2006)!

Theorem (F.–Fusy–Nadeau 2025)

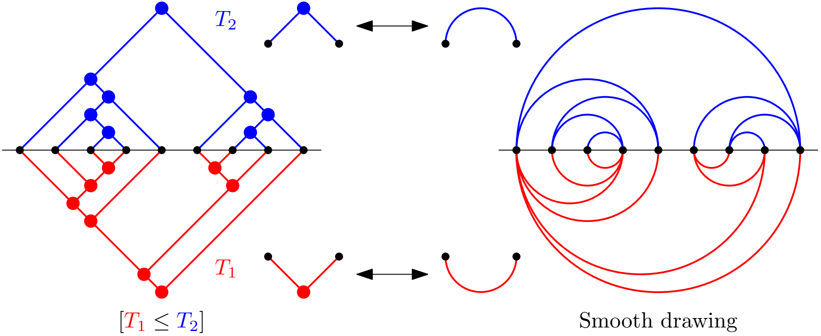
Tamari intervals of size n are in bijection with bicolored blossoming trees with n edges (thus $n+1$ nodes).

Inspired by interval-posets (Châtel–Pons 2015), giving **uniform** enumeration.

Many enumerative and structural consequences.

Canonical drawing and smooth drawing

Canonical drawing: larger tree on top, smaller tree flipped on bottom



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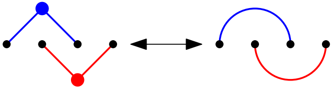


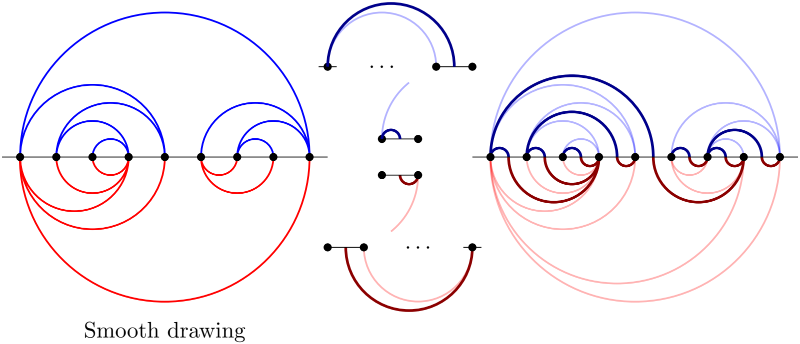
in Tamari interval

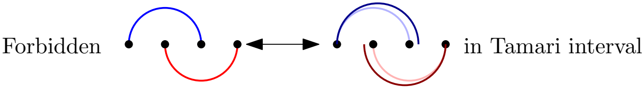
Smooth drawing: replace wedges by semi-circles

Each leaf has arcs of each color in **one direction**, due to **type**.

Smooth drawing and blossoming tree

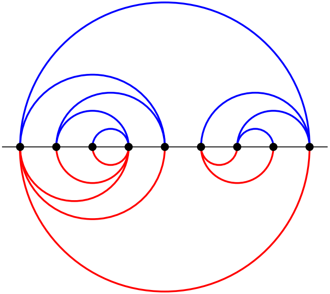
To blossoming tree: each segment draws two half-edges



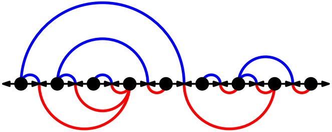


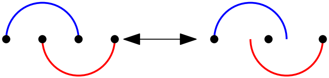
Smooth drawing and blossoming tree

Break the middle line into buds, conditions satisfied!



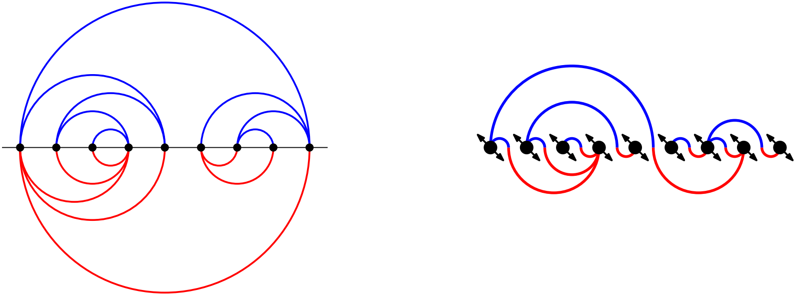
Smooth drawing



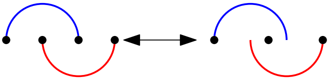


Smooth drawing and blossoming tree

Just wiggle a bit...



Smooth drawing

Forbidden  in Tamari interval

Smooth drawing and blossoming tree

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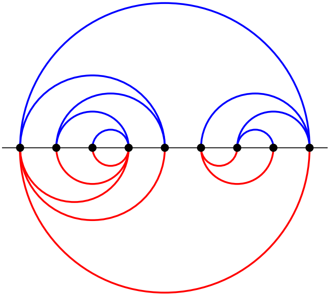
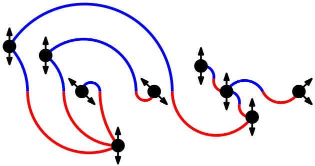
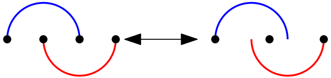


Smooth drawing



Forbidden  in Tamari interval

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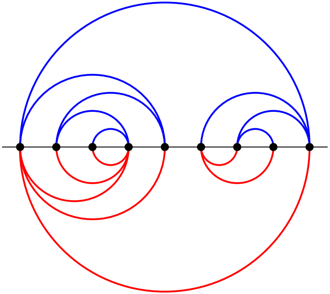
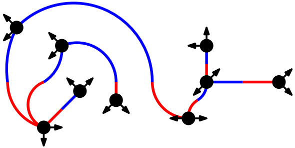
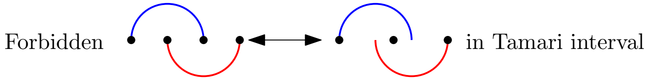


Smooth drawing





Smooth drawing and blossoming tree

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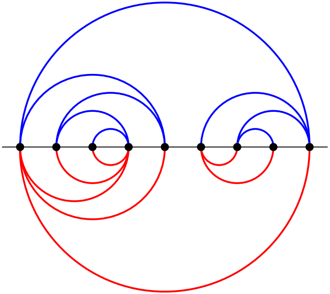

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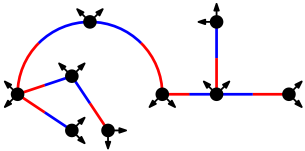

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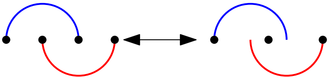
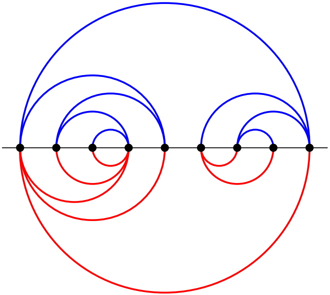


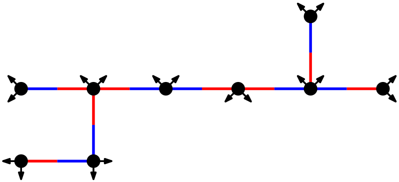
in Tamari interval

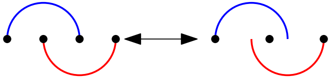
Smooth drawing and blossoming tree

... and we get a nice blossoming tree



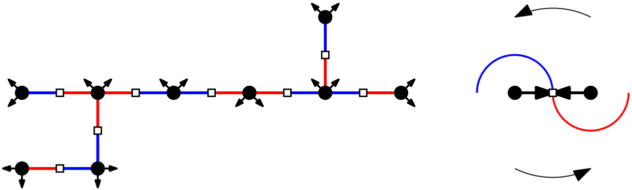
Smooth drawing



Forbidden  in Tamari interval

The reverse direction

To find the order of nodes, we do closure of buds to edges.



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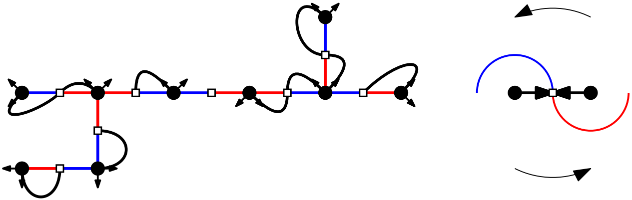

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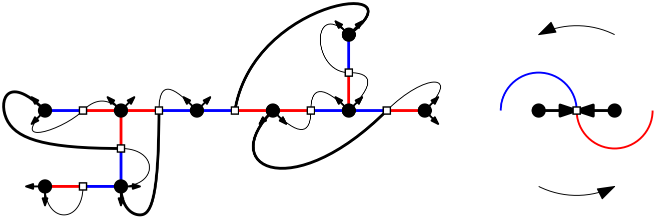

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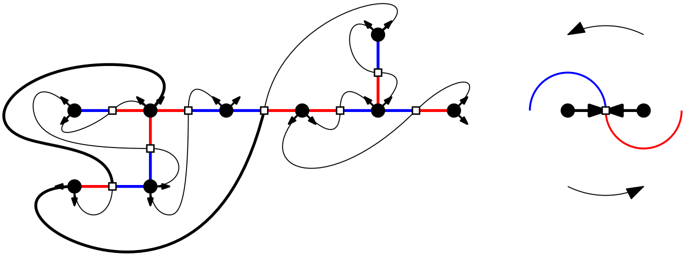
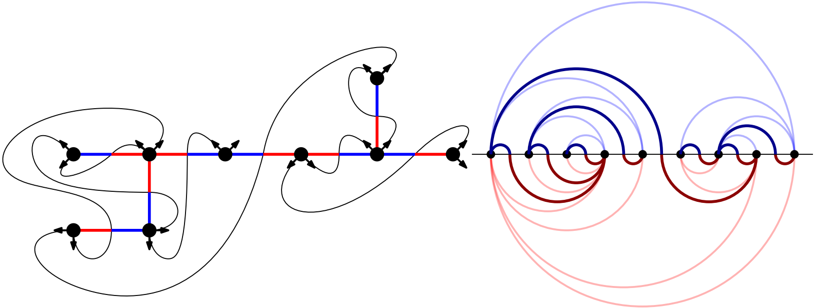


The reverse direction

Stretch the thread, and we get the trees.

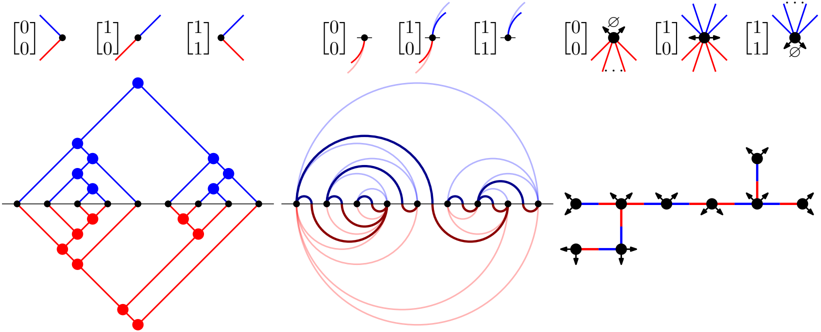


Refined statistics

Type of a leaf: 0 for right child, 1 for left child

Types of a node (pair of leaves): $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$

Statistics considered by Chapoton for new intervals.



Types in blossoming tree: presence of blue/red half-edges

First enumeration result

Theorem (Bostan–Chyzak–Pilaud 2023+)

The number of Tamari intervals of size n with k pairs of leaves of type $\begin{bmatrix}0\\0\end{bmatrix}$ or $\begin{bmatrix}1\\1\end{bmatrix}$ is

$$\frac{2}{n(n+1)}\binom{n+1}{k}\binom{3n}{k-2}.$$

Gives the f-vector of canonical complex of the Tamari lattice!

Synchronized intervals: special case $k = n + 1$

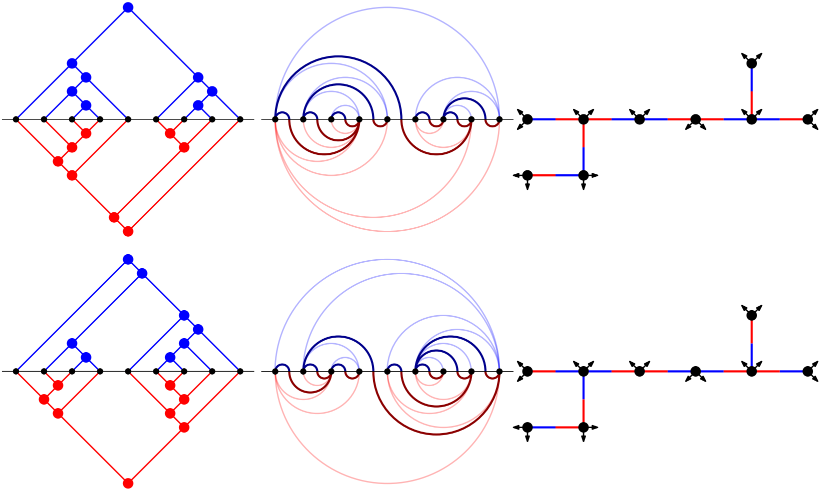
Obtained by solving functional equations.

Blossoming trees: k nodes with adjacent buds among $n + 1$ nodes.

Cyclic lemma suffices!

Duality

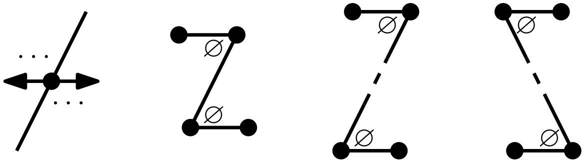
Duality on Tamari intervals: just a half-turn.



Duality on blossoming trees: just exchanging colors.

Different families in patterns

Interesting families can be described by forbidden patterns!



Synchronized Modern Infinitely modern Kreweras

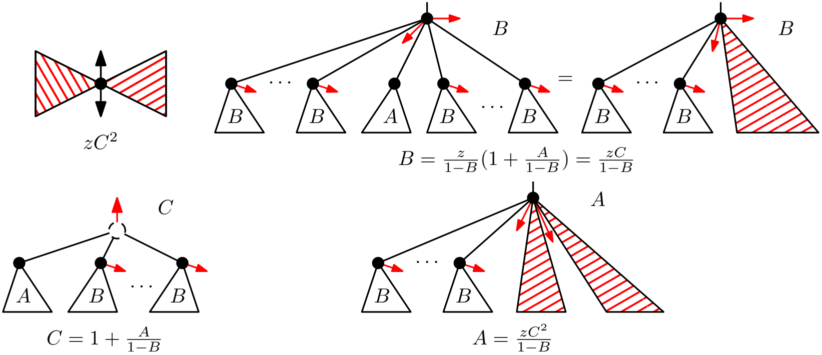
Another proof of bijection in the spirit of (Poulalhon–Schaeffer 2006) for

- General intervals ↔ triangulations (Bernardi–Bonichon 2009)
- Synchronized ↔ non-separable maps (F.–Préville-Ratelle 2017)
- Kreweras ↔ ternary trees (Bernardi–Bonichon 2009)

Unified enumeration

Leads to different tree specifications, thus unified enumeration.

Example: modern intervals, blossoming trees avoiding Z, rooted at a bud



Even with refined by node types and intersection of families!

Self-dual sub-family: those stable by exchanging colors. Doable!

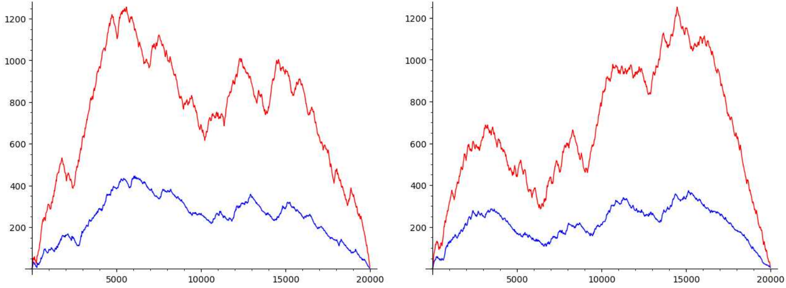
Enumeration results

Types	General size n	Self-dual size $2k$	Self-dual size $2k+1$
General	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	$\frac{1}{3k+1}\binom{4k}{k}$	$\frac{1}{k+1}\binom{4k+2}{k}$
Synchronized	$\frac{2}{n(n+1)}\binom{3n}{n-1}$	0	$\frac{1}{k+1}\binom{3k+1}{k}$
Modern / new for size-1	$\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$	$\frac{2^{k-1}}{k+1}\binom{2k}{k}$	$\frac{2^{k}}{k+1}\binom{2k}{k}$
Modern and synchronized	$\frac{1}{n+1}\binom{2n}{n}$	0	$\frac{1}{k+1}\binom{2k}{k}$
Inf. modern / Kreweras	$\frac{1}{2n+1}\binom{3n}{n}$	$\frac{1}{2k+1}\binom{3k}{k}$	$\frac{1}{k+1}\binom{3k+1}{k}$

Direct combinatorial explanation for many of them, maybe all?

Random generation

- Bijection coded in Sagemath (available on Github)
- Conversion with known structures in Sagemath
- Random generation for blossoming trees in linear time



Random modern intervals of size 100000 in Dyck paths

Discussion

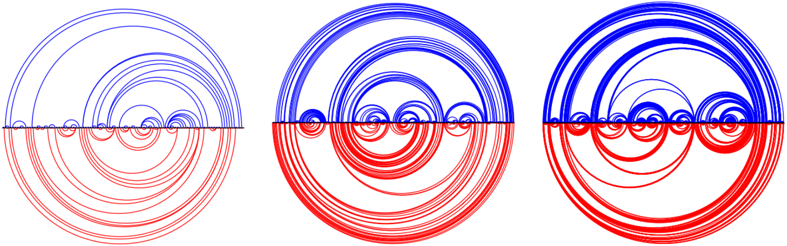
- Quite versatile: solves another family equi-enumerous as Kreweras
- Mysterious involution: reflection on blossoming trees
  - Exchanges infinitely modern and Kreweras
  - What are the images of modern intervals?
- How to explain Reiner's observation:
  self-dual intervals = q-analogue of # general intervals with $q = -1$?
  Works also for synchronized!
- m-Tamari intervals (canopy $(10^m)^n$) have nice formula (Bousquet-Mélou–Fusy–Préville-Ratelle 2011):

$$\frac{m+1}{n(mn+1)}\binom{n(m+1)^2+m}{n-1}.$$

But our bijection breaks the order in canopy, so hard to get it?

Large scale structure?

Maybe related to a recent work of Chapuy on Tamari intervals under the form of Dyck paths?



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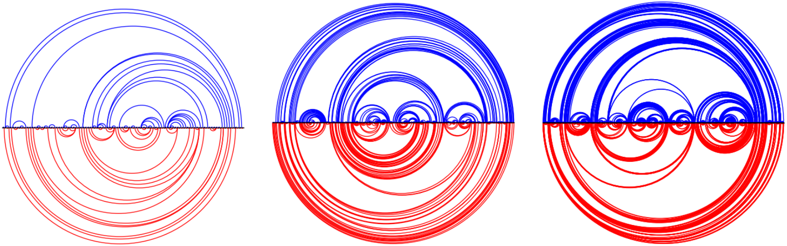
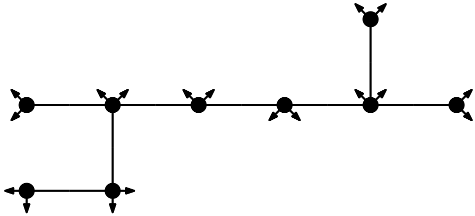


Thank you for listening!

To planar 3-connected triangulations

3-connected triangulation: all faces of degree 3, no loop nor multiple edge.



Each bud connects to the node after two edges, unless blocked by a bud.

To planar 3-connected triangulations

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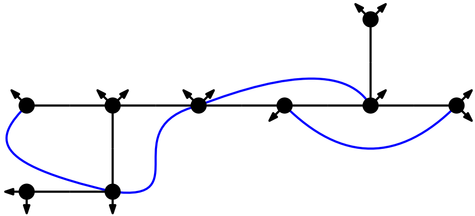

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To planar 3-connected triangulations

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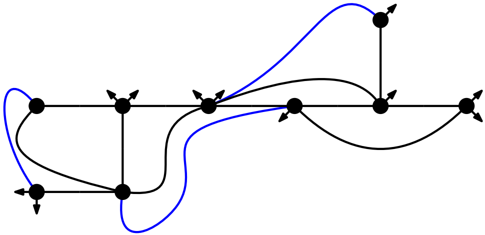
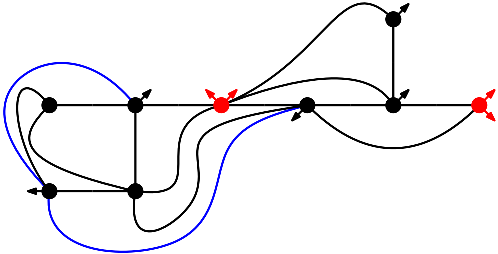


Two nodes left with no match, with two singly-matched paths.

To planar 3-connected triangulations

3-connected triangulation: all faces of degree 3, no loop nor multiple edge.



Put two extra nodes to get a triangulation. $>$ Back $<$