Paths, trees and permutations: some enumerative aspects of Tamari lattices

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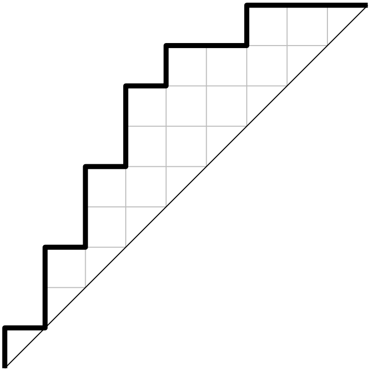
Section 1

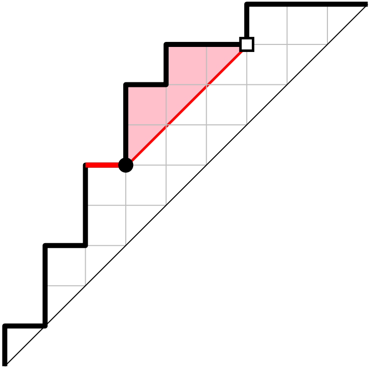
Paths, for introduction

Dyck paths and Tamari lattice, ...



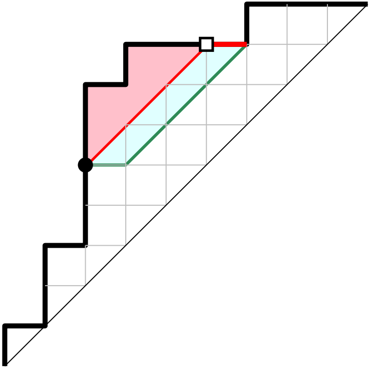
Dyck path: n north(N) and n east(E) steps above the diagonal
Counted by Catalan numbers $\mathrm{Cat}(n) = \frac{1}{2n+1}\binom{2n+1}{n}$

Dyck paths and Tamari lattice, ...



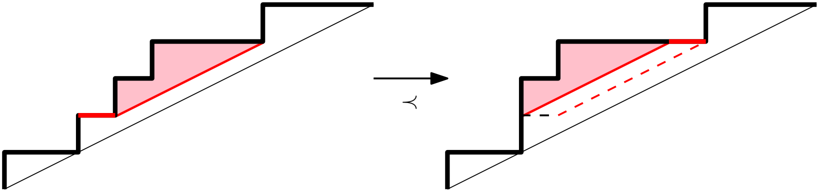
Covering relation: take a valley •, find the next point □ with the same distance to the diagonal ...

Dyck paths and Tamari lattice, ...



... and push the segment to the left. This gives the **Tamari lattice** $\textsc{Tam}(n)$ (Tamari 1962).

..., m-Tamari lattice, ...



m-**ballot** paths: n north steps, mn east steps, above the "m-diagonal".

Counted by Fuss-Catalan numbers $\mathrm{Cat}_m(n) = \frac{1}{mn+1}\binom{mn+1}{n}$.

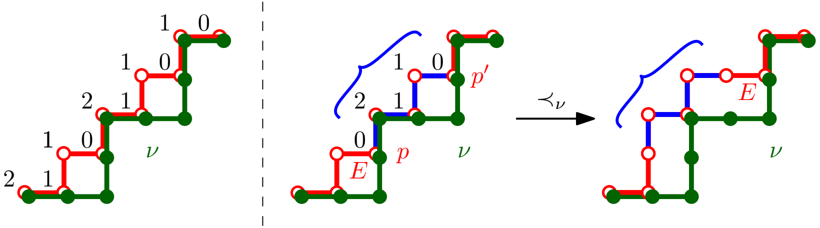
A similar covering relation gives the m-**Tamari lattice** (Bergeron 2010).

Further: **rational Tamari lattice** (Armstrong–Rhoades–Williams 2013)

... and beyond.

But we can use an arbitrary path ν as "diagonal"!

Horizontal distance $=$ $\#$ steps one can go without crossing ν



Generalized Tamari lattice or **ν-Tamari lattice** (Préville-Ratelle and Viennot 2014): $\textsc{Tam}(\nu)$ over arbitrary ν (called the **canopy**).

Partitioning the Tamari lattice by type

Theorem (Préville-Ratelle–Viennot 2014)

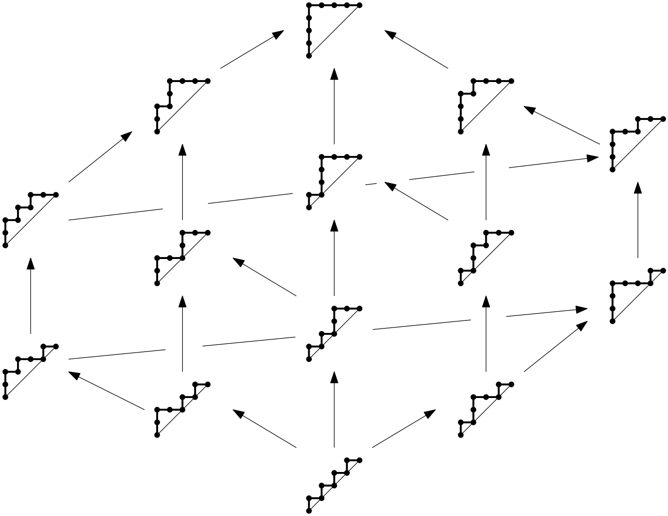
The Tamari lattice of order n is partitioned into 2^{n-1} intervals, each isomorphic to some $\textsc{Tam}(v)$ with v of length $n-1$.

Delest and Viennot (1984): There is a bijection between Dyck path of length $2n$ and an element in $\textsc{Tam}(\nu)$ for some ν of length $n-1$.

Theorem (Préville-Ratelle–Viennot 2014)

$\textsc{Tam}(\nu)$ is isomorphic to the dual of $\textsc{Tam}(\overleftarrow{\nu})$, where $\overleftarrow{\nu}$ is ν reversed with exchange $N \leftrightarrow E$ (flipping the lattice path ν).

Partitionning the Tamari lattice by type



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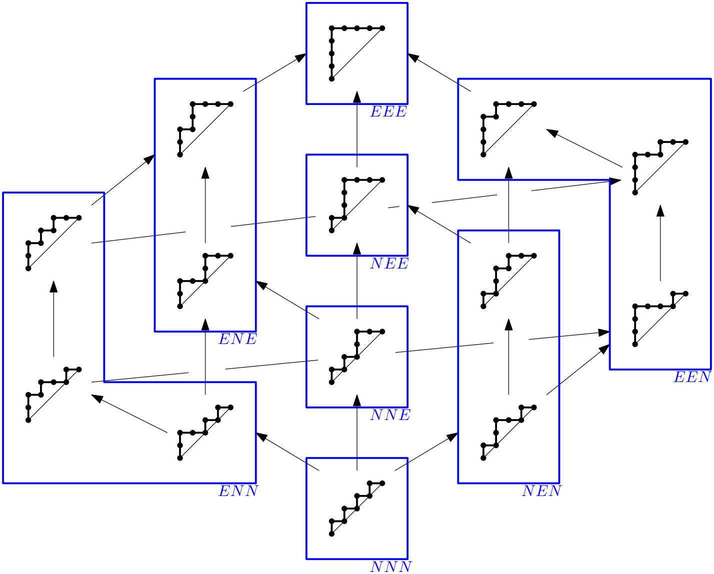

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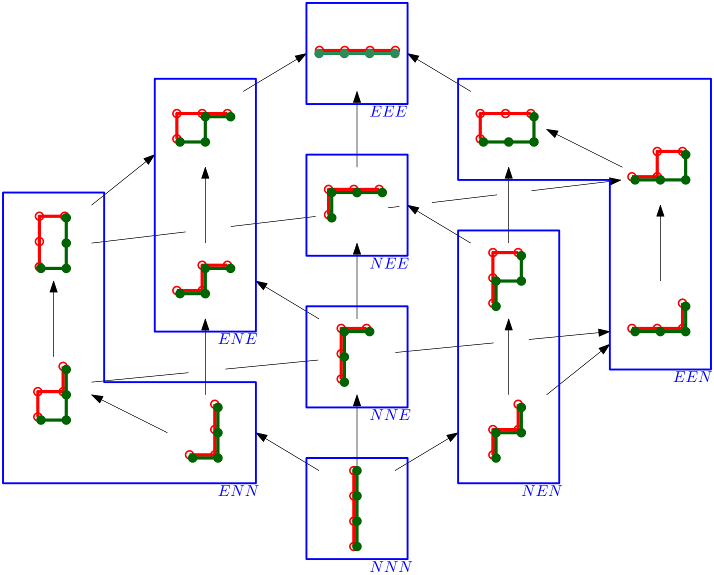


The next level: intervals

Interval: $[a, b]$ with comparable $a \leq b$

Motivation: conjecturally related to trivariate diagonal coinvariant spaces, also with operads... and nice numbers!

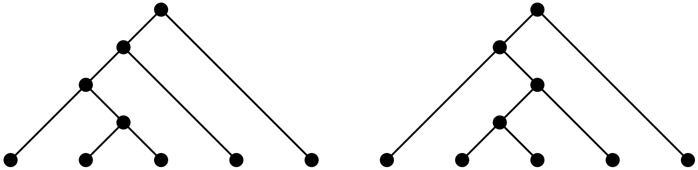
- Counting with functional equations: Bostan, Bousquet-Mélou, Chapoton, Chapuy, Chyzak, Fusy, Pilaud, Préville-Ratelle, ...
- Interval poset: Chapoton, Châtel, Combes, Pons, Rognerud, ...
- Planar maps: Bernardi, Bonichon, Duchi, F., Fusy, Henriet, Humbert, Nadeau, Préville-Ratelle, ...
- λ-terms and proofs: F., N. Zeilberger, ...

Section 2

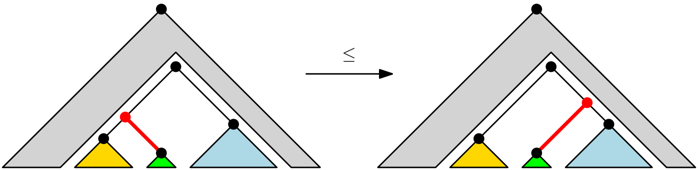
Trees, for bijections

Binary trees

Binary trees : n binary internal nodes and $n+1$ leaves

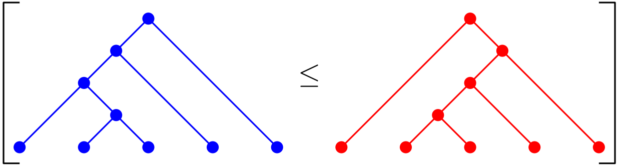


Rotation (from left to right) :



Tamari intervals, with binary trees

An interval $[S, T]$ of binary trees



Bracket vector: size of right sub-tree at each internal node in infix order

$$(1, 0, 0, 0) \leq (3, 0, 0, 0)$$

Also distance of each up step to its matching down step in Dyck paths (Huang–Tamari, 1972). Componentwise order ⇒ Tamari lattice.

Dual bracket vector: size of left sub-tree at each internal node

$$(0, 0, 2, 3) \leq (0, 0, 1, 2)$$

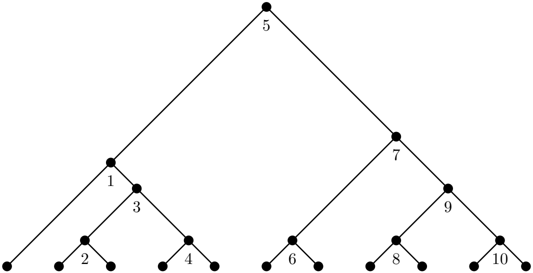
Reversed componentwise order by duality!

Bracket vector, a notion of matching

Bracket vector: $(b_1, \ldots, b_n)$, with b_i the size of right sub-tree of node i

Thus, node i covers $i+1, i+2, \ldots, i+b_i$.

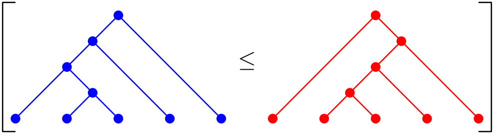
Encoding in parentheses: the "opening" of i is "closed" by $i+b_i$.



$$(3, 0, 1, 0, 5, 0, 3, 0, 1, 0)$$

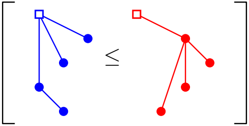
For dual bracket vector $(d_1, \ldots, d_n)$, node i covers $i-1, \ldots, i-d_i$.

Tamari intervals, with plane trees



Binary tree of size $n \Rightarrow$ plane tree with n edges and extra root:

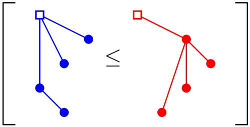
- Remove leaves;
- Left child $\rightarrow$ left sibling;
- Right child $\rightarrow$ rightmost child.



Bracket vector: size of sub-tree of each node in prefix order

The "opening" of a node is "closed" by the last descendant (or itself).

Tamari intervals, three types of bijections



$[S, T]$: Tamari interval with S, T plane trees. Three ways for bijections:

- Take T, and use S as decorations
  - Trees with decorations $\Leftrightarrow$ maps (F.–Préville-Ratelle 2017, ...)
  - Closed flow on forests (Chapoton 2014, Chapoton–Châtel–Pons 2015, F. 2018) or parking on trees (Lackner–Panholzer 2016, Curien–Hénard 2022)
  - Grafting trees (Pons 2019) or $\beta(1,1)$-trees (Cori–Schaeffer 2003)
- Mixing S and T
  - Interval posets (Chapoton–Châtel–Pons 2014, Châtel–Pons 2015)
  - Cubic coordinates (Combe 2023)
  - Blossoming trees (Schaeffer 1997, F.–Fusy–Nadeau 2023+)
- Take S, and use T as decorations
  - Maps with orientation (Bernardi–Bonichon 2009)

Bijections, the first way: closed flow, or parking

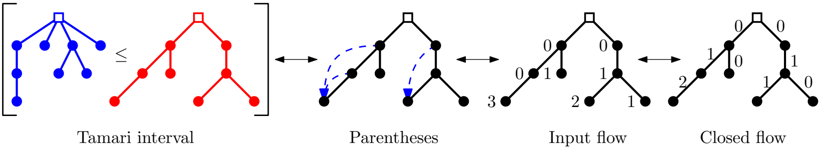
$[S, T]$: Tamari interval with S, T plane trees
$(b_1, \ldots, b_n)$, $(c_1, \ldots, c_n)$: bracket vectors of S, T (resp.). $b_i \leq c_i$ for all i.

"Opening" of node i in S is "closed" by a descendant of i (or itself) in T.

Well-parenthesized: only need #ancestors each node "closes"

Closed flow (Chapoton–Châtel–Pons 2015):

- Root-ward conserved flow, zero at root (well-parenthesized);
- Each non-root node has an input flow (#ancestors "closed");
- Each non-root node consumes 1 unity of flow (its "opening").



Parking on tree (Lackner–Panholzer 2016): input flow of cars parking on nodes

Bijections, the first way: sticky trees

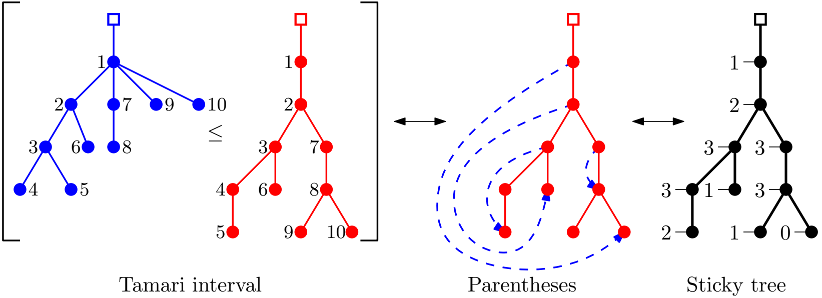
$[S,T]$: Tamari interval with S,T plane trees

"Opening" of node i in S is "closed" by a descendant of i (or itself) in T.

Well-parenthesized: only need the furthest ancestor each node "closes"

Sticky tree (F. 2018): plane tree T with labeling ℓ

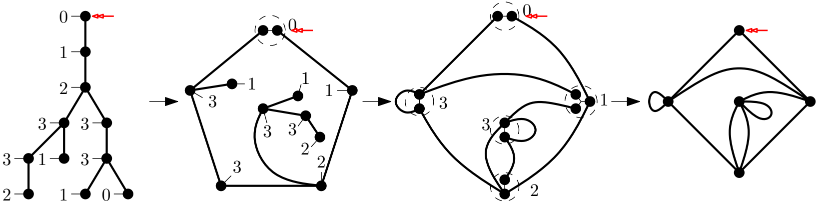
- If u closes no one, then $\ell(u) =$ the depth of u;
- Otherwise, $\ell(u) =$ the depth of the last ancestor it doesn't "close".



Sticky trees and planar maps

Planar map: drawing of graphs on the plane without crossing

Rooted: with one corner of the infinite face distinguished



Sticky trees ⇒ planar bridgeless maps: "sticking" nodes of the same label together, while keeping planarity

Well-parenthesized ⇒ planar and bridgeless

Planar bridgeless maps ⇒ sticky trees: exploration process

Tamari intervals and planar maps

The same approach applies to several families of intervals and maps.

Intervals	Formula	Planar maps
General	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	bridgeless (F. 2018) 3-connected triangulation (Bernardi–Bonichon 2009, F. 2018)
Synchronized	$\frac{2}{n(n+1)}\binom{3n}{n-1}$	non-separable (F.–Préville-Ratelle 2017)
New/modern	$\frac{3 \cdot 2^{n-2}}{n(n+1)}\binom{2n-2}{n-1}$	bipartite (F. 2021)

What we can do with these direct bijections:

- Refined enumeration (*e.g.* (F. 2021))
- Symmetries (*e.g.* (F. 2018, F. 2021))
- Links to other objects (*e.g.*, fighting fish (Duchi–Henrient 2023) and λ-terms (F. 2023))

Bijections, the first way: grafting trees

$[S,T]$: Tamari interval with S,T plane trees

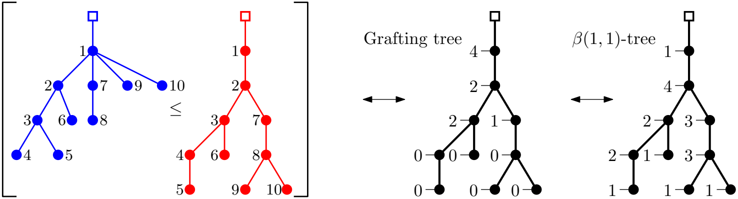
"Closing" of node i in S was "opened" by an ancestor of i (or itself) in T.

Well-parenthesized: simply #children of each node

Grafting tree (Pons 2019), plane tree version: T and labeling ℓ such that

$$\forall \text{ node } u, 0 \leq \ell(u) \leq |T_u| - \textstyle\sum_{v \in T_u \setminus \{u\}} \ell(u).$$

$\ell(u)$: #descendants "opened" by u, or #children of u in S



$\beta(1,1)$-tree (Cori–Schaeffer 2003, decomposition of 3-connected cubic planar maps):
T and labeling ℓ' such that ($\ell'(u)$: #descendants (or itself) not yet "opened")

$$\forall \text{ node } u, 1 \leq \ell'(u) \leq 1 + \textstyle\sum_{v \text{ child of } u} \ell'(u).$$

Bijections, the second way: interval posets

Bracket vector $(b_i)_{i\in[n]}$: **right sub-tree** of node i has $i+1,\dots,i+b_i$

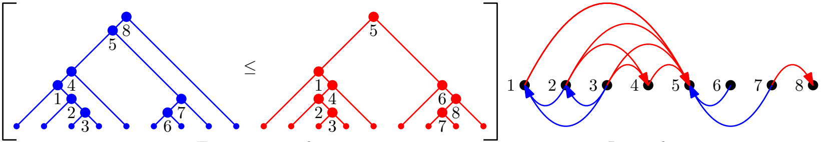
Dual bracket vector $(d_i)_{i\in[n]}$: **left sub-tree** of node i has $i-1,\dots,i-d_i$

$I=[S,T]$: Tamari interval with

- $(b_1,\dots,b_n)$ the bracket vector for S,
- $(d_1,\dots,d_n)$ the dual bracket vector for T.

Interval poset (Chapoton–Châtel–Pons 2014), (Châtel–Pons 2015): $(\le_I,[n])$ with

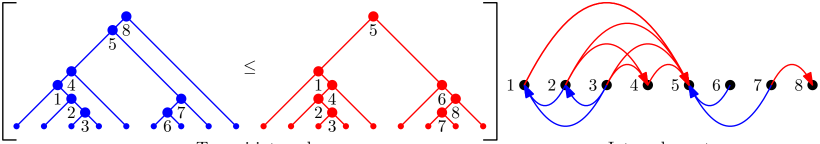
- For all i, we have $i+1,\dots,i+b_i \le_I i$;
- For all i, we have $i-1,\dots,i-d_i \le_I i$.



Tamari interval Interval poset

Tamari condition $\Rightarrow (\le_I,[n])$ goes down on both S,T, thus poset

Applications of interval posets



Tamari interval Interval poset

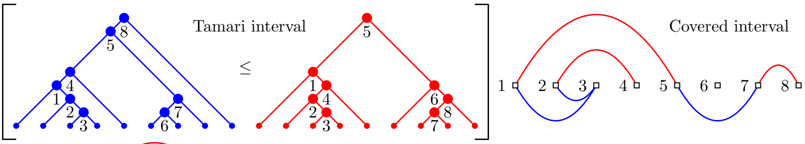
- Rise-contact symmetry of m-Tamari intervals (Pons 2019)
- Study of exceptional and (infinitely) modern intervals (Rognerud 2020)
- Cubic coordinates for geometry of Tamari interval poset (Combe 2023)
- Extended to binary relations on $[n]$ (weak order, Hopf algebra) (Châtel–Pilaud–Pons 2019), (Pilaud–Pons 2020)

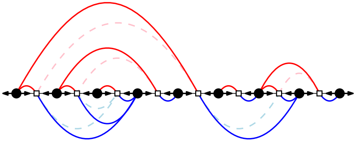
Bijections, the second way: blossoming trees

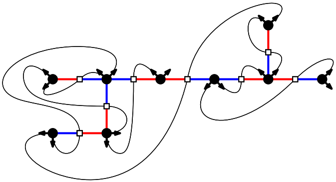
$I = [S, T]$: Tamari interval with

- $(b_1, \ldots, b_n)$ the bracket vector for S,
- $(d_1, \ldots, d_n)$ the dual bracket vector for T,
- Interval poset $(\leq_I, [n])$ with $i - d_i, \ldots, i-1, i, i+1, \ldots, i + b_i \leq_I i$.

Blossoming tree (Schaeffer 1997, F.–Fusy–Nadeau 2023+): 2 buds on each node







Tamari condition $\Rightarrow$ no cycle, thus tree

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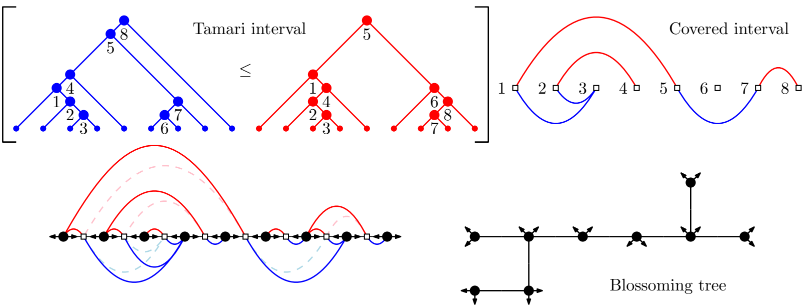

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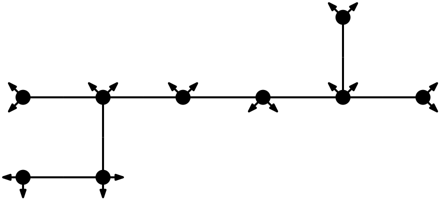
Applications of blossoming trees

Blossoming trees to 3-connected planar triangulations (Poulalhon–Schaeffer 2006), (Albenque–Poulalhon 2013)



- Tree structure ⇒ combinatorial manipulation of generating functions
- Specializes to synchronized and modern intervals, with enumerations
- Easy symmetries (duality ↔ root reversing)
- Combinatorial proof of a formula in (Bostan–Chyzak–Pilaud 2023+)

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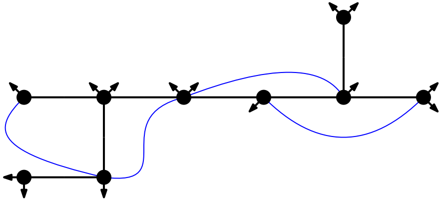

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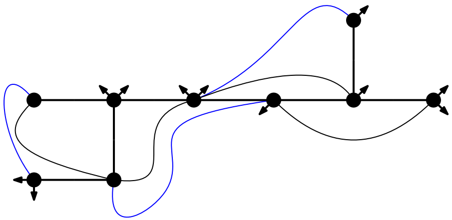

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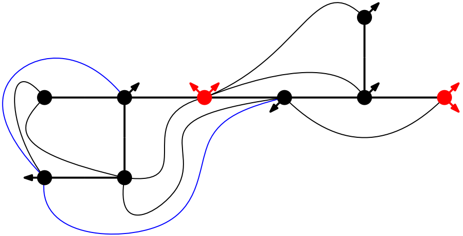

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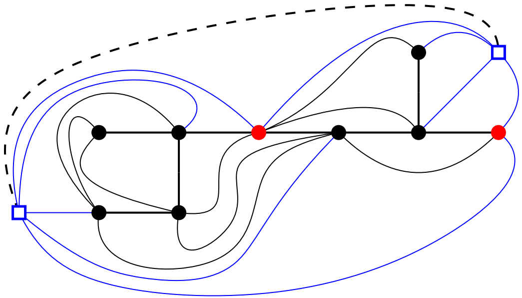

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Bijections, the third way: maps with orientation

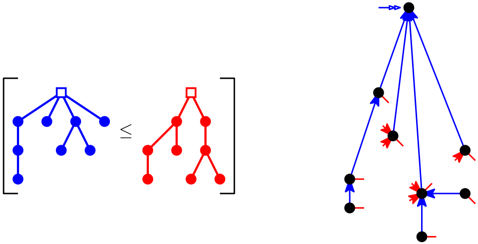
$[S,T]$: Tamari interval with S,T plane trees

"Opening" of node i in T is never (?) "closed" by a descendant of i in S.

Well-parenthesized: "closed" by a child of an ancestor, or the root

Schnyder woods without ccw cycle (Schnyder 1989), (Bernardi–Bonichon 2009):

- Blue tree is S, with red buds given by T for pairing;
- One tail for each node; #heads = #nodes "closed" in T;



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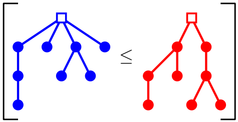

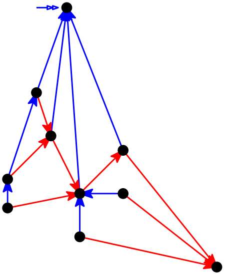

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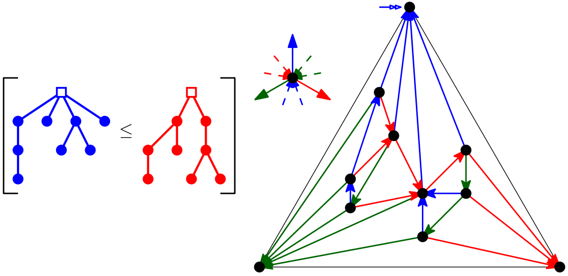

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The case of ν-Tamari

Some of the constructions can be generalized to ν-Tamari

- Equivalent of binary trees and bracket vectors for elements (Ceballos–Padrol–Sarmiento 2020)
- Intervals in bijection with synchronized intervals (F.–Préville-Ratelle 2017), thus bijections with maps (F.–Préville-Ratelle 2017) and fighting fish (Duchi–Henriet 2023)
- Bernardi–Bonichon generalizes to ν-Tamari (Fusy–Humbert 2019+)

Section 3

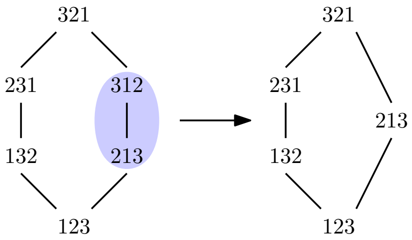
Permutations, for extension

Tamari lattice, as quotient of the weak order

$\mathfrak{S}_n$ as a Coxeter group generated by $s_i = (i, i+1)$

For $w \in \mathfrak{S}_n$, $\ell(w)$ = min. length of factorization of w in s_i

(Left) weak order $\leq_{\text{weak}}$: $s_i w$ covers w iff $\ell(s_i w) = \ell(w) + 1$



Sylvester class : permutations with the same binary search tree

Only one 231-avoiding in each class. Induced order = Tamari.

Generalizing Tamari lattice with Coxeter groups

Coxeter groups: $G = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} \rangle$ with $s_i^2 = 1$ and $m_{i,j} \geq 2$.

Classification: $A_n \cong \mathfrak{S}_{n+1}$, B_n, D_n, $I_2(p)$, $E_6, E_7, E_8, F_4, H_3, H_4$

Cambrian lattices (Reading 2007):

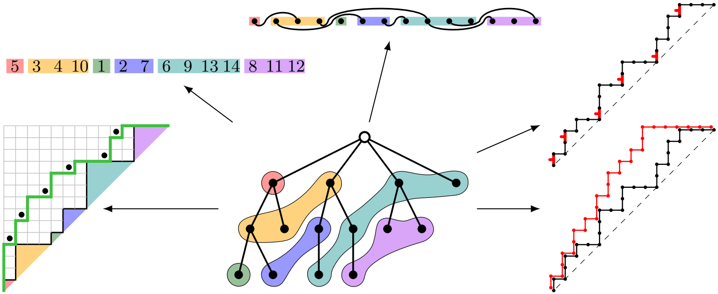
- Works for all types, with combinatorial models;
- On c-aligned elements, with c a Coxeter element (product of all s_i's);
- Different $c \Rightarrow$ same #elements but not for #intervals
- Further generalized to permutrees (Pilaud–Pons 2018) and m-Cambrian lattices (Stump–Thomas–Williams 2018+).

Parabolic Tamari lattices (Mühle–Williams 2019):

- Defined on parabolic quotients: $G^J = G / \langle s_i \mid i \in J \rangle$ for $J \subseteq [n]$
- Generalize Reading's Cambrian construction to parabolic quotients;
- With combinatorial models and bijections for type A.

Parabolic Cataland, type A

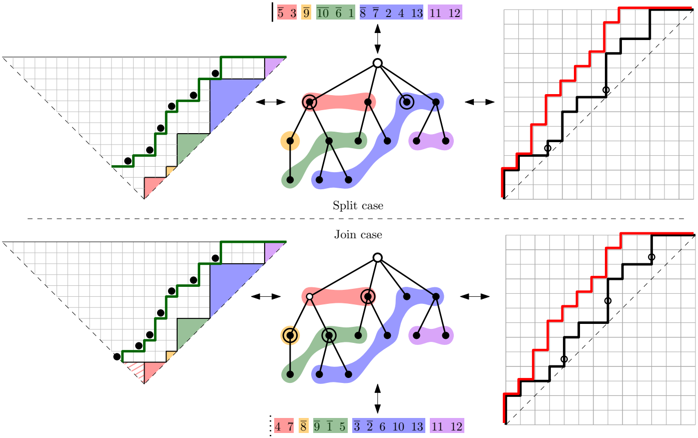
Simple model for parabolic Tamari lattices (Ceballos–F.–Mühle 2020)



- Simplifies some bijections in (Mühle–Williams 2019).
- Isomorphic to certain ν-Tamari lattices.
- Links to walks in the quadrant in (Bousquet-Mélou–Mishna 2010).
- Solves a conjecture in (Bergeron–Ceballos–Pilaud 2022).
- Recovers the zeta map in q, t-Catalan combinatorics.

Parabolic Cataland, type B

Work in progress! (F.–Mühle–Novelli 2023+)



- Counting in special cases
- Recovers the type-C zeta map (Sulzgruber–Thiel 2018)

Open questions

- Find a natural planar map family for (unlabeled) m-Tamari intervals, counted by (Bousquet-Mélou–Fusy–Préville-Ratelle 2011)
$$\frac{m+1}{n(mn+1)}\binom{n(m+1)^2+m}{n-1}.$$
- Bijections between $(m+1)$-constellations and greedy m-Tamari intervals, known (Bousquet-Mélou–Chapoton 2023+) to be counted by
$$\frac{(m+2)(m+1)^{n-1}}{(mn+1)(mn+2)}\binom{(m+1)n}{n}.$$
- Prove that there are as many m-Cambrian intervals as m-Tamari intervals (Préville-Ratelle, personal communication).
- Definition of type-B ν-Tamari lattices, at least for some ν (Ceballos–Padrol–Sarmiento 2020).
- Enumeration of elements in the higher Stasheff–Tamari lattice through maximal chains in Tamari lattice (Rambau 1997, Nelson–Treat 2022).

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Thank you for listening!