Tamari-like intervals and planar maps

Wenjie Fang
TU Graz

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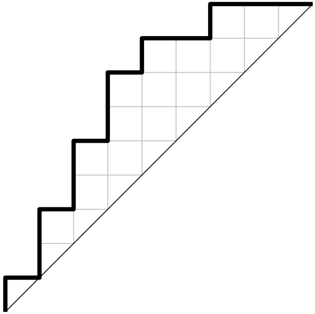
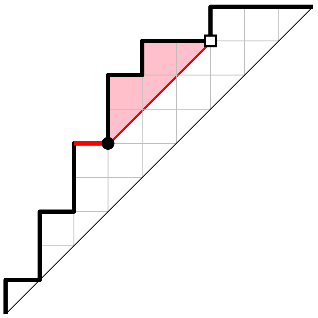
Erwin Schrödinger Institute

Dyck paths and Tamari lattice, ...



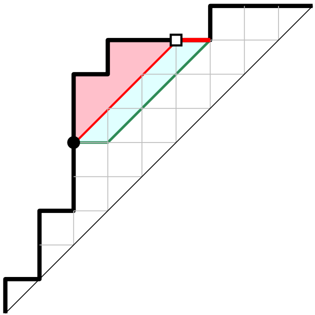
Dyck path: n north(N) and n east(E) steps, always above the diagonal
Counted by the n-th Catalan numbers $\mathrm{Cat}(n) = \frac{1}{2n+1}\binom{2n+1}{n}$

Dyck paths and Tamari lattice, ...



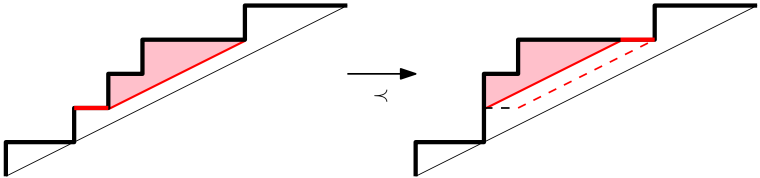
Covering relation: take a valley point •, find the next point □ with the same distance to the diagonal ...

Dyck paths and Tamari lattice, ...



... and push the segment to the left. This gives the **Tamari lattice** (Huang-Tamari 1972).

..., m-Tamari lattice, ...



m-ballot paths: n north steps, mn east steps, above the "m-diagonal".

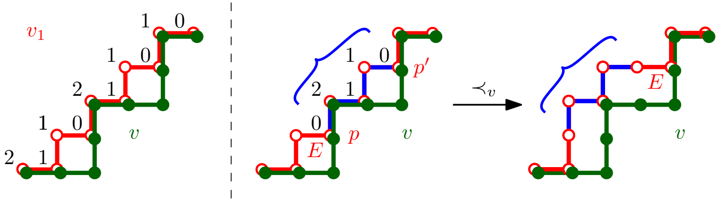
Counted by Fuss-Catalan numbers $\mathrm{Cat}_m(n) = \frac{1}{mn+1}\binom{mn+1}{n}$.

A similar covering relation gives the **m-Tamari lattice** (Bergeron 2010).

… and beyond.

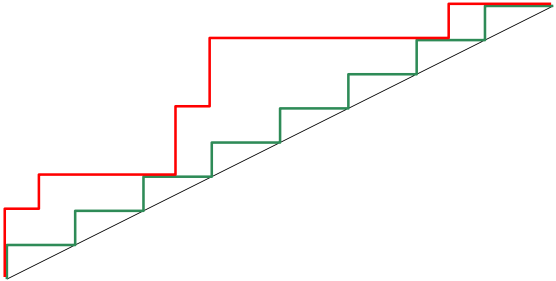
But we can use an arbitrary path v as "diagonal"!

Horizontal distance = # steps one can go without crossing v



Generalized Tamari lattice (Préville-Ratelle and Viennot 2014): $\textsc{Tam}(v)$ over arbitrary v (called the **canopy**) with N, E steps.

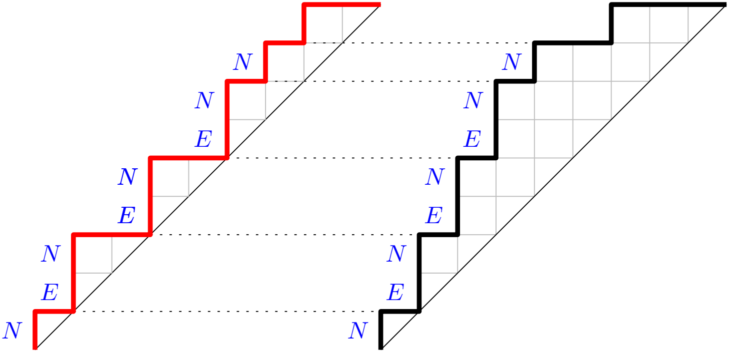
... and beyond.



$\textsc{Tam}((NE^m)^n) \simeq m$-Tamari lattice

Type of a Dyck path

North step: followed by an east step $\rightarrow N$, by a north step $\rightarrow E$.
Mind the change!



Type: $NENENENN$

The two paths have the same type, therefore **synchronized**.

The next level: intervals

Interval in a lattice: $[a, b]$ with comparable $a \leq b$

Motivation: conjecturally related to the dimension of diagonal coinvariant spaces

For generalized Tamari intervals:

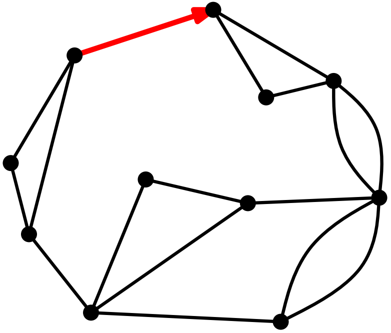
Interval in $\textsc{Tam}(v)$ with v of length $n-1$ $\Leftrightarrow$ **synchronized interval** of length $2n$, *i.e.*, Tamari interval $[D, E]$ with D and E of the same type.

How exactly?

For Tamari and m-Tamari intervals:

- Counting: Bousquet-Mélou, Chapoton, Chapuy, Fusy, Préville-Ratelle, Viennot, ...
- Interval poset: Chapoton, Châtel, Pons, ...
- λ-terms: N. Zeilberger, ...
- Planar maps

What is a planar map?



Planar map: embedding of a connected multigraph on the plane (loops and multiple edges allowed), defined up to homeomorphism, cutting the plane into **faces**

Planar maps are **rooted** at an edge on the infinite outer face.

Intervals that count like planar maps

Chapoton 2006: # intervals in Tamari lattice of size $n =$

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

$=$ # 3-connected planar triangulations with $n+3$ vertices (Tutte 1963)
$=$ # bridgeless planar maps with n edges (Walsh and Lehman 1975)

Bousquet-Mélou, Fusy and Préville-Ratelle 2011:
intervals in m-Tamari lattice of size $n =$

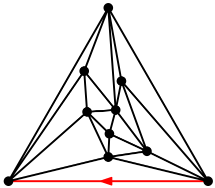
$$\frac{m+1}{n(mn+1)}\binom{n(m+1)^2+m}{n-1},$$

and it also looks like an enumeration of planar maps!

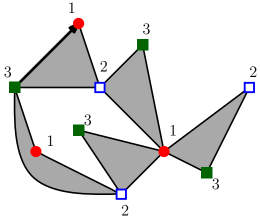
Labeled version: Bousquet-Mélou, Chapuy and Préville-Ratelle 2013

Deeper connections

For Tamari intervals and 3-connected planar triangulations: bijective proof using orientations (Bernardi and Bonichon 2009)

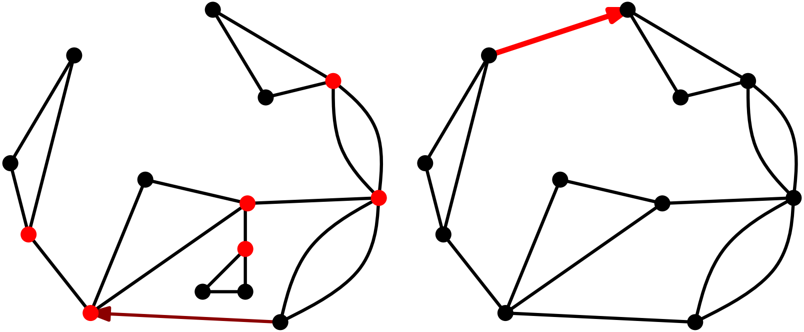


For m-Tamari intervals, the formal method used to solve for its generating function (the "differential-catalytic" method) can also be used on planar m-constellations.



Any other links? Especially for generalized Tamari intervals...

Non-separable planar maps



A **cut vertex** cuts the map into two sets of edges.
A **non-separable planar map** is a planar map without cut vertex.

Another type of intervals that counts like map

Theorem (W.F. and Louis-François Préville-Ratelle 2016)

There is a natural bijection between intervals in $\mathrm{Tam}(v)$ for all possible v of length n and non-separable planar maps with $n+2$ edges.

Intermediate object: **decorated trees**

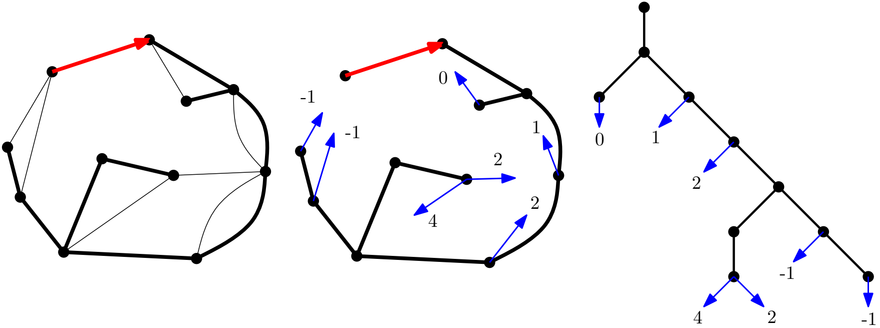
Corollary

The total number of intervals in $\mathrm{Tam}(v)$ for all possible v of length n is

$$\sum_{v \in (N,E)^n} \mathrm{Int}(\mathrm{Tam}(v)) = \frac{2}{(n+1)(n+2)} \binom{3n+3}{n}.$$

This formula was first obtained in (Tutte 1963).

What are decorated trees?



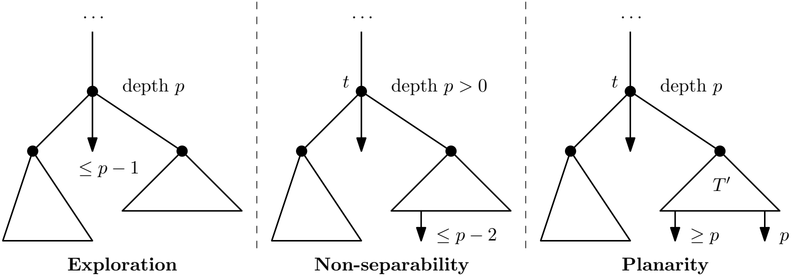
Property

If the exploration of an edge e adjacent to a vertex u reaches an already visited vertex w, then w is an ancestor of u.

Characterizing decorated trees

A **decorated tree** is a rooted plane tree with labels ≥ -1 on leaves such that (depth of the root is 0):

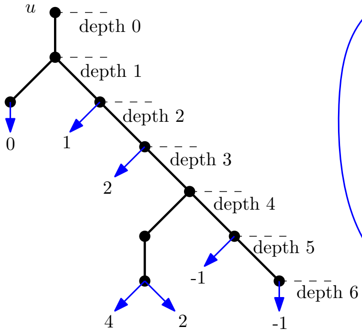
1. (Exploration) For a leaf ℓ of a node of depth p, the label of ℓ is $< p$;
2. (Non-separability) For a non-root node u of depth p, there is at least one descendant leaf with label $\leq p-2$ (the first such leaf is the **certificate** of u);
3. (Planarity) For t a node of depth p and T' a direct subtree of t, if a leaf ℓ in T' is labeled p, every leaf in T' before ℓ has a label $\geq p$.

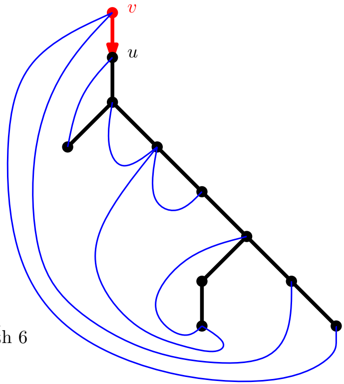


From maps to trees

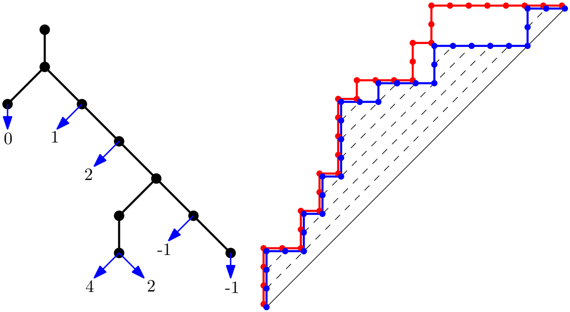
Just glue leaves with label d to their ancestor of depth d.

Only one way to glue back to a planar map.



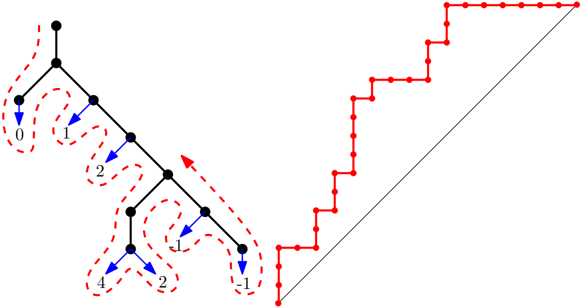


From trees to intervals



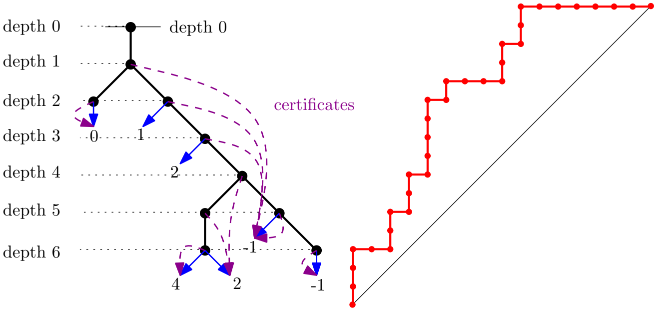
From a decorated tree T to a synchronized interval $[\mathrm{P}(T), \mathrm{Q}(T)]$

From trees to intervals



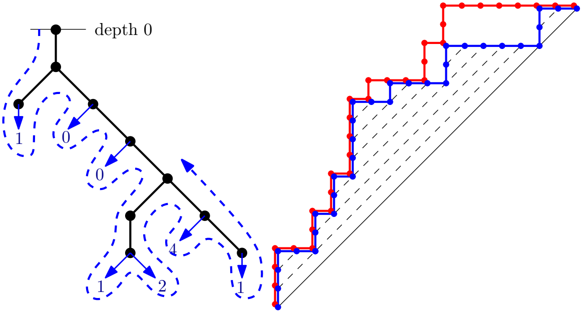
Path Q: a traversal

From trees to intervals



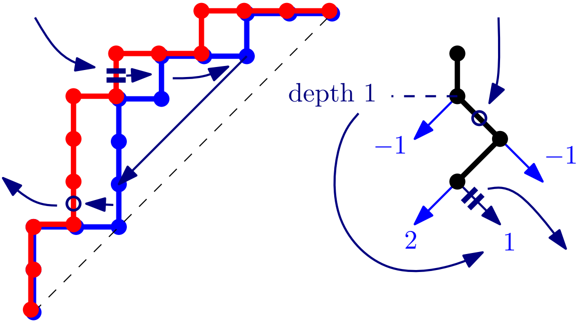
Function c: for a leaf ℓ, $c(\ell) =$#nodes with ℓ as certificate

From trees to intervals

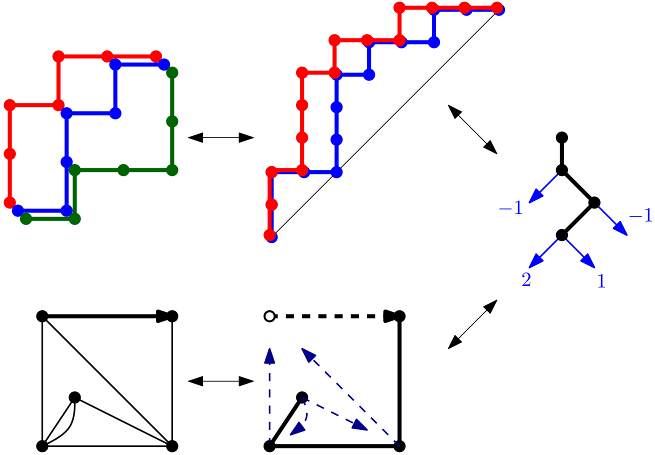


Path P: an altered traversal where descents are $c(\ell) + 1$

The other direction



The whole bijection



Structural result

Our bijections are canonical w.r.t. appropriate recursive decompositions of related objects.

Theorem (W.F. 2017)

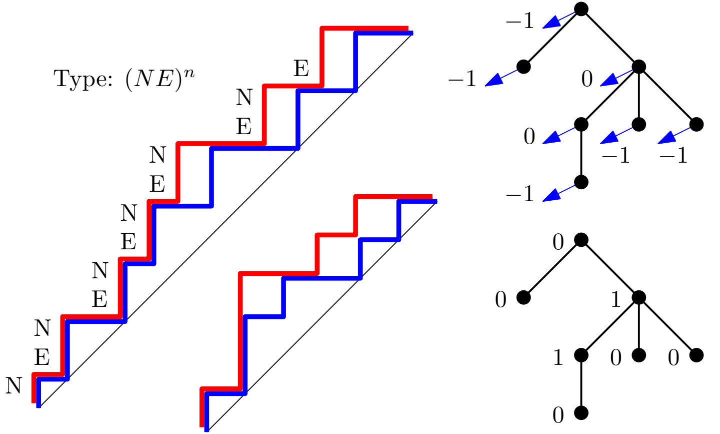
Under our bijections, the involution from intervals in $\textsc{Tam}(v)$ *to those in* $\textsc{Tam}(\overleftarrow{v})$ *is equivalent to map duality.*

Also connection with β-(1,0) trees (Cori, Schaeffer, Jacquard, Kitaev, de Mier, Steingrímsson, ...), leading to a bijective proof of a result in Kitaev–de Mier(2013).

Also equi-distribution results on various statistics

Restriction to the original Tamari intervals...

Tamari lattice = $\textsc{Tam}((NE)^n)$

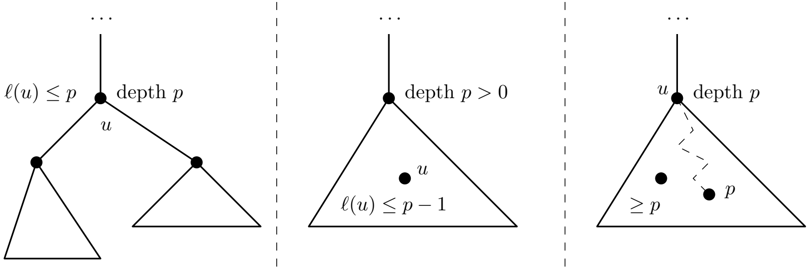


Restriction to type $(NE)^n$: decorated trees where each leaf is the first child of each internal node.

Sticky tree

Decorated trees restricted in $\textsc{Tam}((NE)^n) \rightsquigarrow$ **sticky trees**

A **sticky tree** is a plane tree with a label $\ell(u) \geq 0$ on each node u such that:



Exploration **Absence of bridges** **Planarity**

Essentially adapted from the condition of decorated trees! Now every non-root node has a certificate, which is a node (and can be itself).

Bijections to classical objects

Theorem (W.F. 2017+)

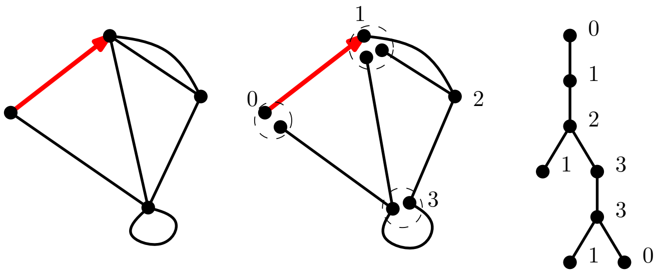
Sticky trees with n edges are in natural bijection with

1. *Tamari intervals with n up steps;*
2. *bridgeless planar maps with n edges;*
3. *3-connected triangulations with $n+3$ vertices.*

A new bijective proof of (1) = (3), different from (Bernardi–Bonichon 2009).

Also a new bijective (and direct!) proof of (2) = (3), different from the recursive ones in (Wormald 1980) and (Fusy 2010).

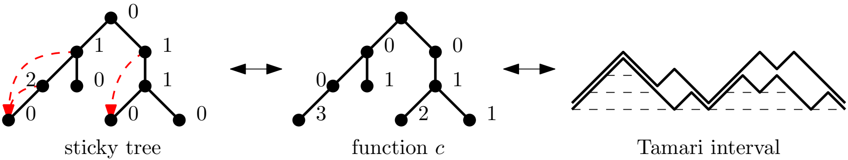
Bijection with bridgeless planar maps



An exploration on edges

There is also a bijection between sticky trees and 3-connected planar triangulations (with a different exploration process)

Bijection with Tamari intervals



Also with closed flows of plane forests (Chapoton–Châtel–Pons 2014), recovering a result therein.

General discussion

- Other related lattices (Stanley, Kreweras, ...) and planar maps (bipartite, constellations)?
- Other structures (*e.g.* 2-stack-sortable permutations)?
- Asymptotic aspects of these objects (statistics, limit shape, ...)?
- Restricted bijections on m-Tamari lattice?

Some interesting sequences...

Number of intervals in $\textsc{Tam}(w^n)$ with w a word in $\{N, E\}$?

Observation

For $w = N^a E N^b$, the number of intervals in $\textsc{Tam}(w^n)$ is of the form

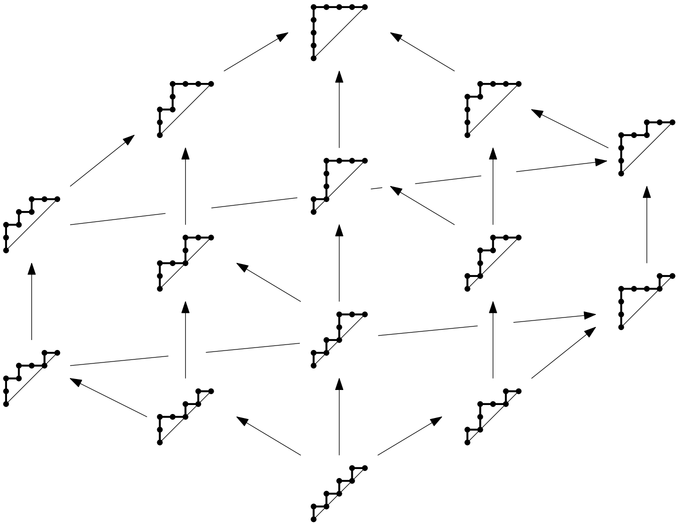
$$\frac{k_{a,b}+1}{n(\ell_{a,b}n+1)}\binom{(a+b+1)^2 n + k_{a,b}}{n-1},$$

where $k_{a,b}$ and $\ell_{a,b}$ are integers. What are these constants?

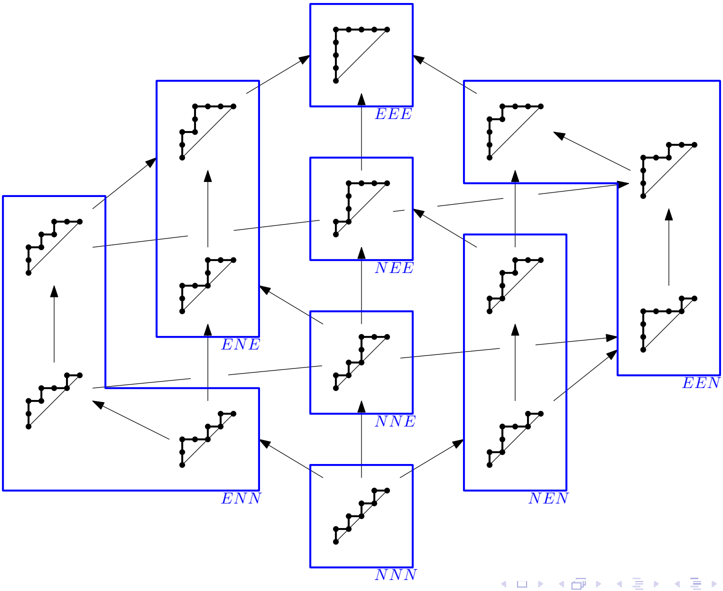
For $w = NNEE$: $1, 20, 755, 37541, 2177653, \ldots$
For $w = NEEN$: $6, 164, 7019, 373358, 22587911, \ldots$

What are these sequences?

Partitionning the Tamari lattice by type



Partitionning the Tamari lattice by type



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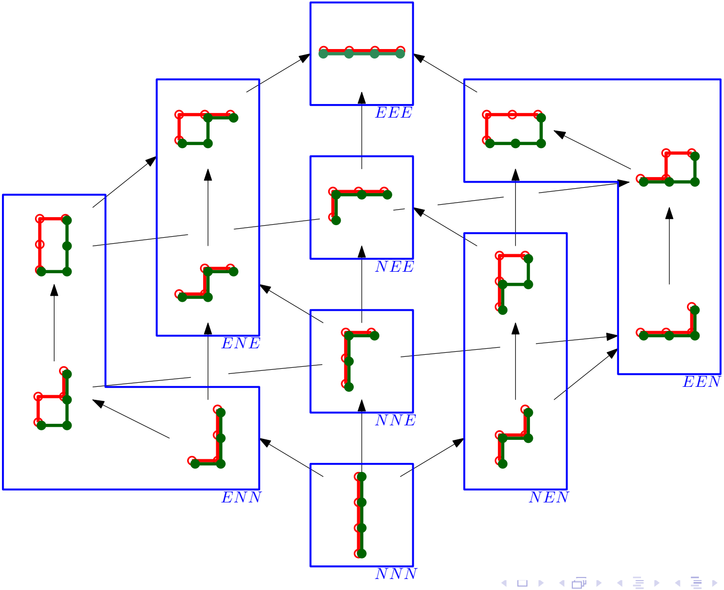

Partitioning the Tamari lattice by type

Delest and Viennot (1984): There is a bijection between Dyck path of length $2n$ and an element in $\textsc{Tam}(v)$ for some v of length $n-1$.

Theorem (Préville-Ratelle and Viennot (2014))

The Tamari lattice of order n is partitioned by path types into 2^{n-1} sublattices, each isomorphic to the generalized Tamari lattice $\textsc{Tam}(v)$ with v the type (a word in N, E of length $n-1$).

Theorem (Préville-Ratelle and Viennot (2014))

The lattice $\textsc{Tam}(v)$ is isomorphic to the order dual of $\textsc{Tam}(\overleftarrow{v})$, where $\overleftarrow{v}$ is the word v read from right to left, with the substitution $N \leftrightarrow E$.

And back...